Analysis of k-out-of-n Redundant Reliability Systems with Multiple Constraints

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Abstract: *In today's competitive environment organizations are being constantly challenged to improve performance and reduce costs. By effectively applying reliability engineering techniques you can achieve dramatic improvements in the reliability and availability of process plant resulting in savings of up to 10% in capital costs and similar savings in plant operation. Generally the reliability of a system is treated as function of cost, but in many real life situations other considerations apart from conventional cost constraint like weight, volume, size, space etc., play vital role in optimizing the system reliability. Quite a few IRM's are reported with cost constraint only in optimizing the system reliability. As the literature informs that few authors mentioned IRM'S with Redundancy and this paper focuses a novel method of optimizing a Redundant IRM with multiple constraints to encounter the hidden impact of additional constraints apart from the cost constraint while the system is optimized by considering the k-out-of-n configuration system.*

Keywords: *IRM, k-out-of-n systems, heuristic, Dynamic, Langrangean and Sensitivity Analysis.*

1. Introduction

The paper is focused to design and to optimize an Integrated Reliability Model for Redundant system with multiple constraints for the k-out-of-n configuration system as a beginning in the mentioned area of the research and initiated optimizing the system reliability. Integrated Reliability Model (IRM)refers to the determination of the number of components (x_j), component reliabilities (r_j), Stage reliabilities (Rj) and the system reliability (R_s) where in the problem considers both the unknowns that is the components reliabilities and the number of components in each stage for the given cost constraint to maximize the system reliability. So far in literature the integrated reliability models are optimized using cost constraint alone where there is an established truth between cost and reliability.

This prompted the author to present a piece of novel aspect of Reliability Optimization through modeling by considering an Integrated Reliability Model for a Redundant System by treating Weight and Volume as additional constraints apart from the conventional Cost constraint to optimize the System Reliability, to negotiate the hidden impact of the additional constraints like Weight and Volume for the k-out-of-n configuration reliability model.

2. Mathematical Model

The objective function and the constraints of the model

$$R_s = \prod_{j=1}^{n} R_j$$

maximize $$R_j = \sum_{k=2}^{x_j} \binom{x_j}{k} . r_j^{\,k} (1 - r_j)^{x_j - k} \quad (1)$$

subject to the constraints

$$\sum_{j=1}^{n} c_j . x_j \leqslant C_0 \quad (2)$$

$$\sum_{j=1}^{n} w_j . x_j \leqslant W_0 \quad (3)$$

$$\sum_{j=1}^{n} v_j . x_j \leqslant V_0 \quad (4)$$

non-negative restriction that x_j is an integer and r_j, $R_j > 0$

3. Mathematical Function

To establish the mathematical model, the most commonly used function is considered for the purpose of reliability design and analysis. The proposed mathematical function

$$c_j = e^{[(1-f_j)(r_j - r_{j,\min})/(r_{j,\max} - r_j)]} \quad (5)$$

Where a_j, b_j are constants.

System reliability for the given function

$$R_s = \prod_{i=1}^{n} R_j \tag{6}$$

The number of components at each stage X_j is given through the relation

$$x_j = \frac{\ln(R_j)}{\ln(r_j)} \tag{7}$$

The problem under consideration is

maximize $$R_s = \prod_{j=1}^{n}[1 \quad (1 \quad r_j)^{x_j}] \tag{8}$$

subject to the constraints

$$\sum_{j-1}^{n} e^{[(1-f_j)(r_j-r_{j,min})/(r_{j,max}-r_j)]}.x_j - C_0 \le 0 \tag{9}$$

$$\sum_{j-1}^{n} e^{[(1-g_j)(r_j-r_{j,min})/(r_{j,max}-r_j)]}.x_j - W_0 \le 0 \tag{10}$$

$$\sum_{j-1}^{n} e^{[(1-h_j)(r_j-r_{j,min})/(r_{j,max}-r_j)]}.x_j - V_0 \le 0 \tag{11}$$

4. The Lagrangian Method:

Solving the proposed formulation using the Lagrangean method

A Lagrangian function is formulated as

$$F = R_s + \lambda_1\left[\left\{\sum_{j=1}^{n}\left\{e^{[(1-f_j)(r_j-r_{j,\min})/(r_{j,\max}-r_j)]}.\frac{\ln(1-R_j)}{\ln(1-r_j)}\right\}\right\} - C_0\right] +$$
$$\lambda_2\left[\left\{\sum_{j=1}^{n}\left\{e^{[(1-g_j)(r_j-r_{j,\min})/(r_{j,\max}-r_j)]}.\frac{\ln 1(-R_j)}{\ln 1(-r_j)}\right\}\right\} - W_0\right] + \lambda_3$$
$$\left[\left\{\sum_{j=1}^{n}\left\{e^{[(1-h_j)(r_j-r_{j,\min})/(r_{j,\max}-r_j)]}.\frac{\ln 1(-R_j)}{\ln 1(-r_j)}\right\}\right\} - V_0\right] \tag{12}$$

The stationary point can be obtained by differentiating the Lagrangean function with respect to R_j, r_j, λ_1, λ_2 and λ_3.

$$\frac{\partial F}{\partial r_j} = \lambda_1\left[\sum_{j=1}^{n} e^{[(1-f_j)(r_j-r_{j,\min})/(r_{j,\max}-r_j)]}.\ln 1(-R_j)/\ln 1(-r_j)\left\{(1-f_j).(r_{j,\max} r_{j,\min}/)\right.\right.$$
$$\left.\left.(r_{j,\max} r_j)^2 + 1/(\ln 1(-r_j)(1-r_j))\right\}\right]$$
$$+\lambda_2\left[\sum_{j=1}^{n} e^{[(1-g_j)(r_j-r_{j,\min})/(r_{j,\max}-r_j)]}.\ln 1(-R_j)/\ln(1-r_j)\left\{(1-g_j).(r_{j,\max} r_{j,\min}/)\right.\right.$$
$$\left.\left.(r_{j,\max} r_j)^2 + 1/(\ln 1(-r_j)(1-r_j))\right\}\right]$$
$$+\lambda_3\left[\sum_{j=1}^{n} e^{[(1-h_j)(r_j-r_{j,\min})/(r_{j,\max}-r_j)]}.\ln 1(-R_j)/\ln(1-r_j)\left\{(1-h_j).(r_{j,\max} r_{j,\min}/)\right.\right.$$
$$\left.\left.(r_{j,\max} r_j)^2 + 1/(\ln 1(-r_j)(1-r_j))\right\}\right] = 0 \tag{13}$$

$$\frac{\partial F}{\partial R_j} = 1 - \lambda_1\left[\left\{\sum_{j=1}^{n} e^{[(1-f_j)(r_j-r_j,\min)/(r_j,\max - r_j)]}\right\}.[1/\ln(1-r_j)][1/(1-R_j)]\right]$$
$$-\lambda_2\left[\left\{\sum_{j=1}^{n} e^{[(1-g_j)(r_j-r_j,\min)/(r_j,\max - r_j)]}\right\}.[1/\ln(1-r_j)][1/(1-R_j)]\right] - \lambda_3$$
$$\left[\left\{\sum_{j=1}^{n} e^{[(1-h_j)(r_j-r_j,\min)/(r_j,\max - r_j)]}\right\}.[1/\ln(1-r_j)][1/(1-R_j)]\right] = 0. \tag{14}$$

$$\frac{\partial F}{\partial \lambda_1} = \left[\left\{\sum_{j=1}^{n}\left\{e^{[(1-f_j)(r_j-r_{j,min})/(r_{j,max}-r_j)]}\right.\right.\right. \tag{15}$$
$$\left.\left.\left..\frac{ln(1-R_j)}{ln(1-r_j)}\right\}\right\} - C_0\right] = 0 \tag{16}$$

$$\frac{\partial F}{\partial \lambda_2} = \left[\left\{\sum_{j=1}^{n}\left\{e^{[(1-g_j)(r_j-r_{j,min})/(r_{j,max}-r_j)]}\right.\right.\right.$$
$$\left.\left.\left..\frac{ln(1-R_j)}{ln(1-r_j)}\right\}\right\} - W_0\right] = 0 \tag{17}$$

$$\frac{\partial F}{\partial \lambda_3} = \left[\left\{\sum_{j=1}^{n}\left\{e^{[(1-h_j)(r_j-r_{j,min})/(r_{j,max}-r_j)]}\right.\right.\right.$$
$$\left.\left.\left..\frac{ln(1-R_j)}{ln(1-r_j)}\right\}\right\} - V_0\right] = 0$$

Where λ_1, λ_2, λ_3 are Lagrangean multipliers.

Case Problem:

To derive the optimum component reliability(r_j), stage reliability(R_j), number of components in each stage(x_j) and the system reliability(R_s) not to exceed system cost Rs.1000, weight of the system 1500 kg and volume of the system 2000 cm^3.

Constants:

Stage	f_j	g_j	h_j	r_j,min	r_j,max
1	0.9	0.5	0.2	0.5	0.99
2	0.9	0.5	0.2	0.5	0.99
3	0.9	0.5	0.2	0.5	0.99

Reliability Design Relating to Cost, Weight and Volume - without X_j Rounding off:

i) Reliability Design Relating to Cost

Stage	r_j	R_j	x_j	$C_{jx}100$	$c_j. x_j$
01	0.701	0.9399	2.33	110.27	256.93
02	0.70	0.9518	2.52	110.00	277.2
03	0.80	0.9886	2.78	110.51	307.21
Total Cost					**841.34**

ii) Reliability Design Relating to Weight

Stage	r_j	R_j	X_j	W_jx100	$W_j. X_j$
01	0.701	0.9399	2.33	163.08	379.97
02	0.70	0.9518	2.52	161.6	407.23
03	0.80	0.9886	2.78	164.8	458.14
Total Weight					**1245.34**

iii) Reliability Design Relating to Volume

Stage	r_j	R_j	X_j	$V_{jx}100$	$V_j .X_j.$
01	0.701	0.9399	2.33	218.7	509.57
02	0.70	0.9518	2.52	215.5	543.06
03	0.80	0.9886	2.78	222.6	618.83
Total Volume					**1671.46**

System Reliability= Rs = 0.8844

Reliability Design Relating to Cost, Weight and Volume - with X_j Rounding off:
Reliability Design Relating to Cost

Stage	r_j	R_j	x_j	$c_{jx}100$	$c_j. x_j$
01	0.701	0.9732	3	110.27	330.31
02	0.70	0.9730	3	110.0	330.00
03	0.80	0.9920	3	110.51	331.53
Total Cost					**991.84**

Variation in Total Cost = 18.82%

Reliability Design Relating to Weight

Stage	r_j	R_j	X_j	$W_{jx}100$	$W_j. X_j$
01	0.701	0.9732	3	163.08	489.24
02	0.70	0.9730	3	161.6	484.8
03	0.80	0.9920	3	164.8	494.4
Total Weight					**1468.44**

Variation in Total Weight = 18.83%

Reliability Design Relating to Volume

Stage	r_j	R_j	X_j	$V_{jx}100$	$V_j .X_j.$
01	0.701	0.9732	3	218.7	656.1
02	0.70	0.9730	3	215.5	646.5
03	0.80	0.9920	3	222.6	667.8
Total Volume					**1970.4**

Variation in Total Volume = 18.79%

System Reliability = 0.9394

Variation in System Reliability = 06.22%

5. Heuristic Method

The Lagrangean multipliers method gives a solution to arrive at an optimal design quickly rather than sophisticated algorithms. This is of course done at the cost of treating the number of components in each stage (x_j) as real. This disadvantage can be overcome, by the heuristic approach. Heuristic methods, in most cases employ experimentation and trial-and-error techniques. A heuristic method is particularly used to come rapidly to a solution that is reasonably close to the best possible answer, or 'optimal solution'.

i) Reliability Design Relating to Cost

Stage	r_j	R_j	x_j	$c_{jx}100$	$c_j.\ x_j$
01	0.701	0.9732	3	110.27	330.31
02	0.70	0.9730	3	110.0	330.00
03	0.80	0.9600	2	110.51	221.02
Total Cost					**881.33**

Variation in Total Cost = 11.86%

ii) Reliability Design Relating to Weight

Stage	r_j	R_j	X_j	$W_{jx}100$	$W_j.\ X_j$
01	0.701	0.9732	3	163.08	489.24
02	0.70	0.9730	3	161.6	484.8
03	0.80	0.9600	2	164.8	329.60
Total Weight					**1303.64**

Variation in Total Weight = 13.09%

iii) Reliability Design Relating to Volume

Stage	r_j	R_j	X_j	$V_{jx}100$	$V_j\ .X_j.$
01	0.701	0.9732	3	218.7	656.1
02	0.70	0.9730	3	215.5	646.5
03	0.80	0.9600	2	222.6	445.2
Total Volume					**1747.8**

System Reliability (Rs) = 0.9091

Variation in Total Volume = 12.61%

Variation in System Reliability = 02.79%

6. Sensitivity Analysis

It is observed that when the input data of constraints is increased by 10% there is only a 4.09% increase in system reliability. When the input data is decreased 10%, there is only an 8.3% decrease in system reliability. When one factor is varied, keeping all the other factors unchanged, the variation in the system reliability is as shown in the following Table.

Variation in factors		System Reliability
Cost	10% decrease	No change
	10% Increase	No change
Weight	10% decrease	No change
	10% Increase	No change
Volume	10% decrease	8.37% decreases
	10% Increase	4.09% increase

The analysis confirms that the volume factor is more sensitive to input data than are cost and weight.

7. Dynamic Programming

The heuristic approach commonly provides a workable solution which is approximate one. To validate the established redundant reliability system and to obtain the much needed integer solution the Dynamic Programming method is applied. The Lagrangean Method can be used as the input for the Dynamic

Programming Approach, in order to determine the stage Reliabilities, System Reliabilities, Stage Cost and the System Cost. The Dynamic Programming Approach provides flexibility in determining the number of components in each stage; Stage Reliabilities and the System Reliability for the given System Cost. As per the procedure the parameter values derived from the Lagrangean are given as inputs for the Dynamic Programming Approach to obtain the integer solution.

i) Reliability Design Relating to Cost Constraint

Stage	r_j	R_j	X_j	C_j	$C_j.\ X_j$
01	0.9075	0.9794	1	1920	1920
02	0.9277	0.9811	1	1590	1590
03	0.9278	0.9416	1	878	878
Total Cost					**4388**

Variation in Total Cost = 12.24%

ii) Reliability Design Relating to Weight Constraint

Stage	r_j	R_j	X_j	W_j	$W_j.\ X_j$
01	0.9075	0.9794	1	2560	2560
02	0.9277	0.9811	1	2385	2385
03	0.9278	0.9416	1	1676	1676
Total weight					**6621**

Variation in Total Weight = 11.72%

iii) Reliability Design Relating to Volume:

Stage	r_j	R_j	X_j	V_j	$V_j\ .X_j.$
01	0.9075	0.9794	1	1920	1920
02	0.9277	0.9811	1	1590	1590
03	0.9278	0.9416	1	878	878
Total volume					**4388**

System Reliability (RS) = 0.9047

Variation in Total Volume = 12.24%

Variation in System Reliability = 19.42%

8. Conclusions

The integrated reliability models for redundant systems with multiple constraints for the k-out-of-n configuration system is established for the commonly used mathematical function using Lagrangean method approach where component reliabilities (r_j) and the number of components (x_j) in each stage are treated as unknowns. The system reliability (R_s) is maximized for the given cost, weight and volume by determining the component reliabilities (r_j) and the number of components required for each stage (x_j). The Lagrangean Multiplier Method provide a real valued solution, the Heuristic approach is considered for analysis purpose which provided a near optimum solution wherein the values of component reliabilities (r_j) are taken as input to carry out heuristic analysis. The analysis of Heuristic approach results in gaining a solution which ought to be an approximate one even after its validation and to derive the much needed scientific integer solutions for the defined problem, the Dynamic Programming approach is applied. The advantage of Dynamic Programming is that the number of components required for each stage (x_j) directly gives an integer value along with the other values of the parameters, which is very convenient for practical implementation for the real life problems.

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