Total Domination Number of Cartesian Product of Cycles $C_m \square C_n$ For m = 6, 7

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Abstract: *Let G = (V, E) be a graph. A set of vertices $S \subseteq V$ is called a total dominating set of G if every vertex of G is adjacent to some vertex in S. The total domination number γ_t (G) of a graph G is the cardinality of a minimum total dominating set. In this paper, we enumerate the method of constructing a minimum total dominating set and thereby determine the total domination number γ_t ($C_m \square C_n$) for cartesian product of cycles $C_m \square C_n$ for any n and m = AMS 6 and 7.*

Subject Classification: *05C69, 05C38*

Keywords: *Cartesian Product, Total dominating set, Total domination number.*

I. INTRODUCTION

Let G = (V, E) be a simple graph (i.e. a graph with neither loops nor multiple edges) with the vertex set V and edge set E. We follow Bondy and Murty's book [1] for teminologies and notations. If u, v $\in$ V then the edge joining u and v is denoted by u v. Any two vertices u, v $\in$ V are said to be adjacent if u v $\in$ E. For a vertex v $\in$ V, the set of vertices that are adjacent to v is said to be an open neighbourhood of v, denoted by N (v). The closed neighborhood of v, denoted by N [v] = N (v) $\cup$ {v}. For a subset S of V, we define the open neighborhood of S as N (S) = $U_{v \in s}$ (N (v). The closed neighbourhood of S is denoted by N [S] and is defined as N [S] = S $\cup$ N(S). A set D $\subseteq$ V of vertices in a graph G = (V, E) is a dominating set if N [S] = V. The domination number γ (G) of a grph G is the cardinality of a minimum dominating set in G. The subject D $\subseteq$ V is called a total dominating set of G (this concept was introduced by Cockayne et al. in [3]) if N (S) = V. The total domination number is the cardinality of a minimum total dominating set in G.

In this paper we study the total domination number of cartesian product of cycles. The cartesian product of two graphs G_1 and G_2 is defined as the graph $G_1 \square G_2$ with vertex set V (G_1) × V (G_2) and (u_1, u_2) (v_1, v_2) $\in$ E (G_1, G_2) if either [$u_1 = v_1$ and u_2 v_2 $\in$ E(G_2)] or [$u_2 = v_2$ and u1v1 $\in$ E (G1)]. Problems of determining various domination parameters for the cartesian product of graphs were intensively studied by many since Vizing posed the conjucture that "the domination number of the cartesian product of any two graphs is at least as large as the product of their domination number". This conjecture is still open.

Cartesian product of two paths is called a complete grid graphs, where as that of cycles is called a toroidal grid graph. The problem of determining the domination numbers of cartesian products of particular graphs is a difficult one. Domination number of cartesian product of paths were intensively studied ([2] and [5]). Domination number of cycles were determined in [4] and [7]. Total domination number of cartesian product of complete grid graphs were calculated in [5] and [8].

We denote by P_n, C_n and K_n respectively the path, cycle and complete graph on n vertices. For convenient we use 1, 2...., n to denote the vertices of C_n. Thus {(i, j): i = 1, 2, ..., m and j = 1, 2,...., n} represent the set of all vertices of $C_m \square C_n$. For j $\in$ V (C_n), we define the set $(C_m)_j$ = C_m {j} and for i $\in$ V (C_m), i (C_n) =

$\{i\}\square C_n$. Thus $(C_m)_j$ and $_i(C_n)$ represent the jth column and ith row of $C_m \square C_n$ respectively. Our strategy for establishing is as follows:

Given by $C_m \square C_n$ can be split in to a primary block Q_1 which contain q ($\geq$ 1) number of $m\square t$ blocks (where t is to be determined $1 \leq t \leq n$ and = $\lfloor \frac{n}{t} \rfloor$) and a secondary block Q_2 a $m\square r$ block, where r = 0, 1,...., t-1. Each $m\square t$ block is called a base block. This base block is constructed by the enumeration process such that the total dominating set formed is a minimum one. We construct total dominating set as such overlapping (i.e. no vertex is totally dominated by more than one vertex) is very less. Another important thing to be noted is when we repeat the pattern of above formed total dominating set in the following base blocks of Q_1 then the induced subgraph of the resulting total dominating set should contain no isolated vertex.

II. KNOWN RESULTS

In [9], we have obtained the following on the total domination number of $C_m\square C_n$ for m = 1 to 5 and for any n.

Proposition 2.1 [9] For any $n \geq 3$, $\gamma_t(K_1\square C_n)$ =

$$\begin{cases} \frac{n}{2}+1 \text{ if } n \equiv 2 \pmod 4 \\ \lceil \frac{n}{2} \rceil, \text{ otherwise.} \end{cases}$$

Proposition 2.2 [9] For any $n \geq 3$, $\gamma_t(P_2\square C_n)$ =

$$\begin{cases} \frac{2n}{3}+1 \text{ if } n \equiv 4 \pmod 6 \\ \lceil \frac{2n}{3} \rceil, \text{ otherwise.} \end{cases}$$

Proposition 2.3 [9] For any $n \geq 3$, $\gamma_t(C_3\square C_n)$ = $\lceil \frac{4n}{5} \rceil$

Proposition 2.4 [9] For any $n \geq 4$, $\gamma_t(C_4\square C_n)$ =

$$\begin{cases} n, & \text{if } n \equiv 0 \pmod 4 \\ n+1, & \text{if } n \equiv 1,3 \pmod 4 \\ n+2, & \text{otherwise.} \end{cases}$$

Proposition 2.5 [9] For any $n \geq 5$, $\gamma_t(C_5\square C_n)$ =

$$\begin{cases} \lceil \frac{4n}{3} \rceil & \text{if } n \equiv 1,2,5 \pmod 6 \\ \frac{4n}{3} & \text{if } n \equiv 0,3 \pmod 6 \\ \lfloor \frac{4n}{3} \rfloor & \text{if } n \equiv 4 \pmod 6 \end{cases}$$

III. MAIN RESULTS

We now determine total domination number of $C_m\square C_n$ for any n and m = 6, 7 as follows:

$$\gamma_t(C_5\square C_n) = \begin{cases} \lceil \frac{5n}{3} \rceil & \text{if } n \equiv 0, 1, 2, 3 \pmod 6 \\ \lceil \frac{5n}{3} \rceil & \text{otherwise} \end{cases}$$

Proof: Let D be a total dominating set for the base block $6\square t$ where t is to be determined. Suppose we start with (1,1) in we have to include one vertex among {(1,2), (2,1), (1, t), (6, 1) to retain the characteristic of total domination. Here we include the vertex (6, 1). With {(1,1), (6,1)} we can totally dominate the vertices (1,2), (1,t), (2,1), (5,1), (6,2) and (6, t) of the base block. We can't include (3,1) and (4,1) in the first column to D, since their inclusion cause overlapping.

In the next step we include (3,2) and (4,2) in our total dominating set. From {(1,1), (6,1), (3,2), (4,2) we can totally dominate (1,2), (2,1), (1,t), (5,1), (6,2), (6,t), (3,1), (3,3), (2,2), (4,1), (4,3) and (5,2). Inclusion of any vertex in the third column leads to overlapping so we skip it. From the fourth we have to include (1,4),(2,4),(5,4) and (6,4) to cover the remaining vertices in previous columns. In the next step if we include (3,6) and (4,6) in our total dominating set, we can totally dominate all the vertices of $6\square 6$. Thus by enumeration process we have determined $6\square 6$ as our base block with a total dominating set ((1,4), (1,4), (2,4), (3,2), (3,6), (4,2), (4,6), (5,4), (6,1), (6,4) for $C_6\square C_n$. Hence given any $C_6\square C_n$ can be split in to a primary block Q_1 and if necessary a secondary block Q_2. The primary block Q_1 contain $\lfloor \frac{n}{6} \rfloor$ number of $6\square 6$ (base) blocks and the secondary block Q_2 contain $r\square 6$ block, where r = 1, 2, 4, 5. The vertices of the primary block Q_1 is totally dominated by the vertices of the set as shown in Fig. 1.

S = $\{(1,1 + 6k), (1,4 + 6k), (2,4 + 6k), (3,2 + 6k), (4,2 + 6k), (3,6 + 6k), (4,6 + 6k), (5,4 + 6k), (6,1 + 6k), \frac{6,4 + 6k}{k} = 0,1, ..., \lfloor \frac{n-1}{6} \rfloor\}$.

If $n \equiv 0 \pmod 6$ then $C_6\square C_n$ is totally dominated by the set S. Thus $\gamma_t(C_6\square C_n) \equiv \lceil \frac{5n}{3} \rceil$. If n 1,2,4,5 (mod 6), then primary block Q_1 is dominated by the set S and secondary block Q_2 is totally dominated by the set isomorphic to R_1, R_2, R_4 and R_5 respectively as shown in Fig. 2.

When $n \equiv 3 \pmod 6$ then $C_6\square C_n$ is split into $\frac{n}{3}$ number of $6\square 3$ block. Each $6\square 3$ block is totally dominated by the set isomorphic to R_3. Thus $\gamma_t(C_6\square C_n)$ = $\frac{5\times n}{3} = \lceil \frac{5n}{3} \rceil$. In the case of $n \equiv 1 \pmod 6$ is split in to (which consist of $\lfloor \frac{n}{3} \rfloor$ number of $6\square 6$ blocks) and a secondary block $Q_2 \simeq R_1$. We can totally dominate $C_6\square C_n$ by S and a set isomorphic to R_1. Thus we have

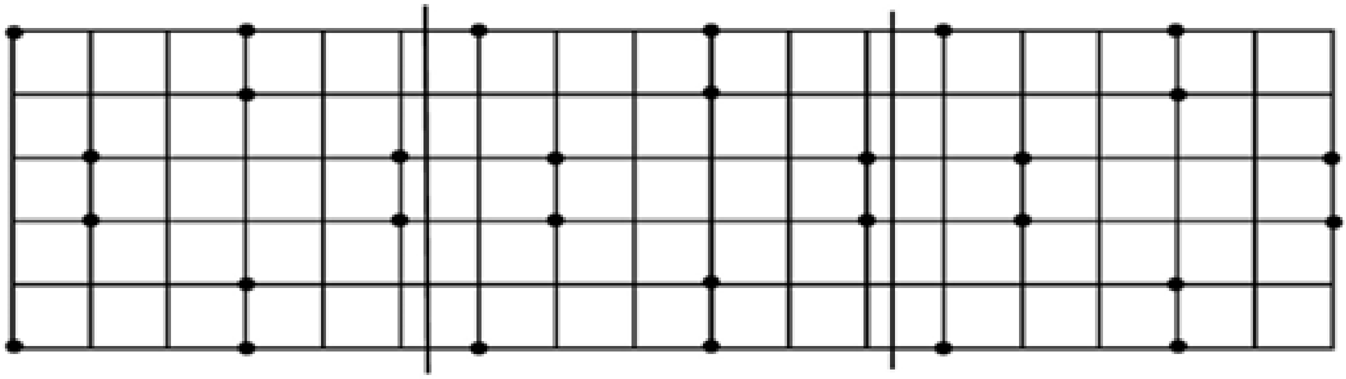

Fig. 1: Total dominating set for the primary block Q_1 of $C_6\Box C_n$

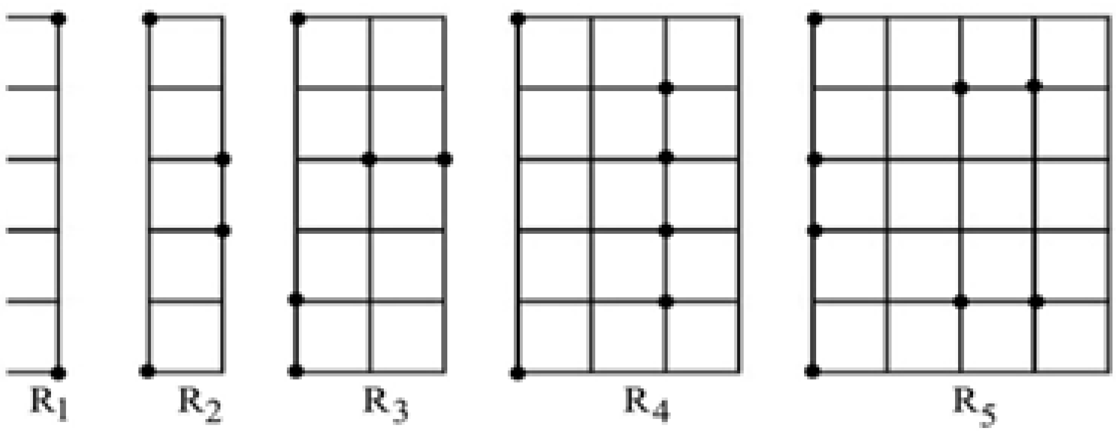

Fig. 2: Total dominating set for the secondary block Q_2 of $C_6\Box C_n$

$\gamma_t\,(C_6\Box C_n) = \frac{10\ (n-1)}{6} + 2 = \left\lceil\frac{5n}{3}\right\rceil$. Similarly for $n \equiv 2 \pmod 6$ we can prove that $\gamma_t\,(C_6\Box C_n) = \left\lceil\frac{5n}{3}\right\rceil$. If $n \equiv 4 \pmod 6$ then n columns of $C_6\Box C_n$ is divided into $\left(\frac{n-4}{4}\right)$ number of $6\Box 6$ blocks (primary blocks Q_1) and a $6\Box 4$ block (secondary block Q_2). Each $6\Box 6$ (base block) is totally dominated by 10 vertices and $6\Box 4$ is totally dominated by 6 vertices. Thus $\gamma_t\,(C_6\Box C_n) = \frac{10 \times (n-4)}{6} + 6 = \left\lfloor\frac{5n}{3}\right\rfloor$.

Note: When n is a multiple of 5 then base block of $C_6\Box C_n$ constructed in the above theorem is not a minimum one. Thus we construct another base block for this particular case.

Theorem 3.2 If n is multiple of 5 then

$\gamma_t\,(C_6\Box C_n) = \frac{8n}{5}$.

Proof: By enumeration process we find that is $6\Box 5$ is the base block for $C_6\Box C_n$, where n is a multiple of 5. Each base block $6\Box 5$ is totally dominated by the set {(1,1), (3,1), (4,1), (6,1), (2,3) (2,4), (5,3), (5,4)}. Thus a given $C_6\Box C_n$ can be split number $\frac{n}{5}$ number of blocks. $6\Box 5$ blocks. And it is totally dominated by the set S = {(1,1 + 5k), (3,1 + 5k), (4,1 + 5k), (6,1 + 5k), (2,3 + 5k) (2,4 + 5k), (5,3 + 5k), (5,4 + 5k)/k = 0,1,... $\frac{n}{5}$ }

Hence we have $\gamma_t\,({}_6\Box C_n) = \frac{8n}{5}$.

Theorem 3.3 For any $n \geq 4$, $\gamma_t\ (C_7\Box C_n) =$

$$\begin{cases} \left\lceil\frac{9n}{5}\right\rceil & \text{if } n \equiv 0,4 \pmod 6 \\ \left\lceil\frac{9n}{5}\right\rceil + 1 & \text{if } n \equiv 1 \pmod 5 \\ \left\lceil\frac{9(n+1)}{5}\right\rceil & \text{otherwise.} \end{cases}$$

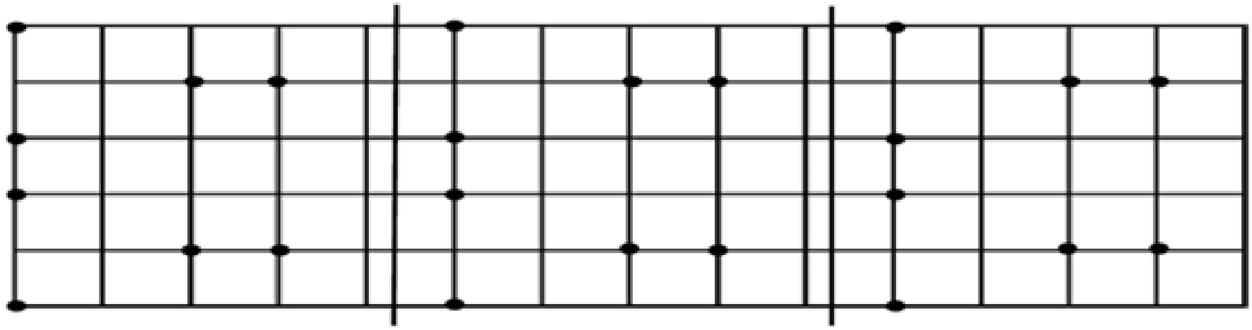

Fig. 3: Total dominating set of $C_6 \square C_n$

Proof: We construct the total dominating set for $C_7 \square C_n$ as follows: By enumeration process we have determined the base block for $C_7 \square C_n$ to be $7 \square 5$ with a total dominating set {(1,1), (1,5), (3,2), (3,3), (3,4), (5,1), (5,5), (6,3), (7,3)}. Thus given any $C_7 \square C_n$ be split in to a primary block Q_1 and a secondary block Q_2. The primary block Q_1 contain $\frac{n}{7}$ number of $7 \square 5$ and Q_2 contain a $7 \square r$, where r = 1 to 4. The primary block Q_1 is totally dominated by the set $S = \{(1,1 + 5k), (1,5 + 5k), (3,2 + 5k), (3,3 + 5k), (3,4 + 5k), (5,1 + 5k), (5,5 + 5k), (6,3 + 5k), (7,3 + 5k)/k = 0,1, \ldots .. \lfloor \frac{n-1}{5} \rfloor \}$.

Suppose $n \equiv 0 \pmod 5$ then there is no secondary region. $C_7 \square C_n$ is totally dominated by S itself. Thus $\gamma_t (C_7 \square C_n) = \frac{9}{5}$. If $n \equiv 1 \pmod 5$, primary region Q_1 is totally dominated by S, whereas secondary region Q_2 is totally by the set isomorphic to R_1. Thus we have $\gamma_t (C_7 \square C_n) = 9\left(\frac{n-1}{5}\right) + 3 = \left\lceil \frac{9n}{5} \right\rceil$. Similarly in the case of $n \equiv 2,3,4$ primary region Q_1 is totally dominated by S and that of secondary region Q_2 is totally dominated by the set isomorphic to R_2, R_3 and R_4 respectively.

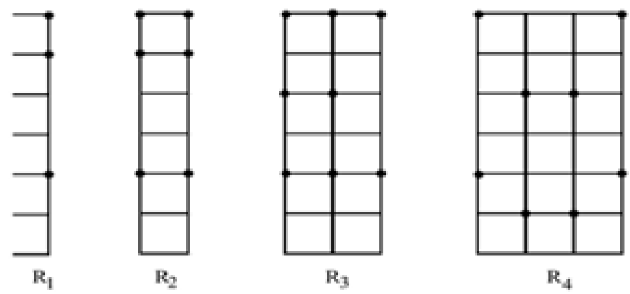

Fig. 5: Total dominating set for the secondary Q_2 of $C_7 \square C_n$

If $n \equiv 2 \pmod 5$ then each base block $7 \square 5$ is totally dominated by nine vertices and secondary block $7 \square 2$ is totally dominated by set isomorphic to R_2. Thus we calculate the total domination number for $C_7 \square C_n$ as $\gamma_t (C_7 \square C_n) = 9 \left(\frac{n-1}{5}\right) + 6 = \left\lceil \frac{9(n+1)}{5} \right\rceil$. Similarly we can prove for $n \equiv 3 \pmod 5$. Suppose $n \equiv 4 \pmod 5$ then primary region of Q_1 is totally dominated by S and secondary region Q_2 ($7 \square 4$ block) is done by eight vertices. Hence. we get $\gamma_t (C_7 \square C_n) = \frac{9(n-4)}{5} + 8 = \left\lceil \frac{9n}{5} \right\rceil$. This completes the proof.

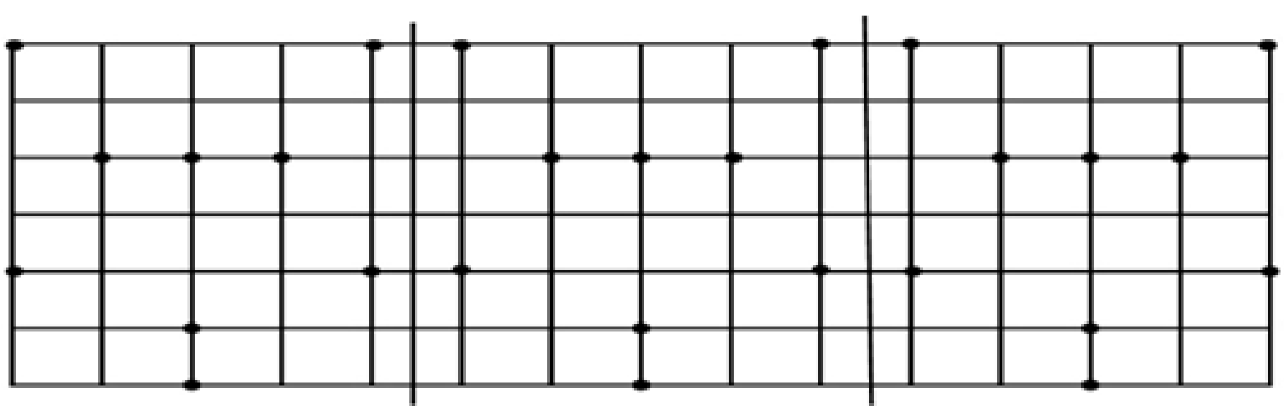

Fig. 4: Total dominating set for primary block Q_1 of $C_7 \square C_n$

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