Change Point Method with Weibull Distribution

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Abstract: *The assumption of known in-control mean and SD underlies in all the standard charting methods (Shewart, CUSUM, and EWMA) and change point approach. The values used are generally not exact parameter values, but estimates are obtained in this paper. Using estimated parameters, the ARL behaviour changes randomly from one realization to another, making it impossible to control run length behaviour of any particular chart. The unknown – parameter change point formulation methodology for detecting and diagnosing step changes based on imperfect process knowledge is studied. It is observed that, despite not requiring specification of the post-change process parameter values, its performance is never far short of that of the optimal CUSUM chart which requires this knowledge, and it is far superior for shifts away from the CUSUM shift for which the CUSUM chart is optimal. Also, we observe change point methods are designed for step changes that persist, they are also competitive with the Shewart chart.*

Key words: *Cumulative Sum Control Charts, Exponentially Weighted Moving Average Control Charts, Shewart Control Charts, Average Run Length, Statistical Process Control.*

I. INTRODUCTION

The Statistical Process Control aims to detect and diagnose situations where the process gone out of statistical control. This type of problem involves two aspects namely process and statistical aspects to have detailed outline of some of its statistical modelling aspects see Crowder et al [1]. In fact the status of statistical quality control may be described as one in which the process readings appear to follow a common statistical model. One model is that the process is in SQC, the successive process readings X_i's are independent and sampled from the same distribution. When the process goes out of control, it may behave in different ways. In general we can have two types of causes, those that affect single process readings and then disappear, and the second type of causes or sustainable causes. These causes will continue until they are identified and eliminated. In statistical terminology the isolated causes analogous to an out layer. The Shewart $\overline{X}$ with an R or s-chart is an excellent tool for detecting these special causes, namely isolated or sustainable causes. However Shewart control char is less effective for detecting small changes in the process. Standard tools for detecting sustain changes are the CUSUM and EWMA chart. This paper focuses on another, less familiar method aimed at detecting sustained changes is the Change Point formulation when the process averages follow non Normal distribution.

II. CHANGE POINT METHOD

At first let us discuss Change Point Method when the process readings modelled by two Normal Distributions,

$$\left.\begin{array}{l} X_i \sim N(\mu_1, \sigma_1^2) \text{ for } i = 1, 2\ldots \ldots .\ n. \\ \qquad\qquad \square \\ X_i \sim N(\mu_2, \sigma_1^2) \text{ for } i = \tau + 1, \ldots\ n. \end{array}\right\} \qquad (1)$$

Here the number of observations 'n' is fixed in any traditional statistical settings, but this 'n' may increase unlimitedly in phase – II Statistical Process Control settings. Here both settings i.e. Phase-I and Phase-II will be discussed, with context indicating which of the two applies. In the case of in control distribution is N $(\mu_{1,}\ \sigma_1{}^2)$, the readings follows this distribution up to an epoch τ, the change point, at which point they shift to another Normal distribution different in mean i.e. $\mu_1 = \mu_2$, in variability $\sigma_1 = \sigma_2$ or in both mean and variance. In change point method process readings leads to two statistical tasks namely testing task and estimating task. The testing task is to conform whether there has indeed being a change. If the change is there then the task of estimating ô, the time at which shift occurred. Sometimes we may have to estimate some or all of the parameters μ_1, μ_2, σ_1 and σ_2. In all discussions [see Hawkins, 2003] we work on change point has focused on shifts in mean only. i.e. $\mu_1 \neq \mu_2$ but $\mu_1 = \mu_2 = \sigma$, throughout this paper this type of frame work is used here.

From this change point method we have three scenarios based on the amount of process knowledge viz.,

1. All parameters μ_1, μ_2, σ_1 and σ_2 are known exactly priory.
2. In control parameters μ and σ are known but μ_2 is unknown.
3. All parameters μ_1, μ_2, σ_1 and σ_2 are unknown.

In this paper, we concentrate on the third scenario namely all parameters are unknown under varying distributions.

III. IMPLEMENTATION OF CHANGE POINT METHOD

The formulas below mentioned indicate that the $T_{max,n}$ values are computationally burdensome. For the implementation of the method, construct two arrays of values of

$$S_n = \sum_{i=1}^{n} X_i \text{ and } \qquad W_n = \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$$

there is no need to store the running mean $\overline{X}_n = \frac{S_n}{n}$, but it will be calculated 'on the fly'. The observation of the two new table entries can be calculated from the numerically stable recursions, when a new observation is added.

$$S_n = S_{n-1} + X_n \text{ and } W_n = W_{n-1} + \frac{[(n-1) X_n - S_{n-1}]^2}{n(n-1)}$$

The two sample statistic T_{jn} are calculate for every possible split point $1 \leq j \leq n$ after finding $T_{max,n}$. It is easy and more covenient to find T^2_{jn}. The variance explained by a split at point j can be shown to be

$$E_{jn} = \frac{(nS_j - jS_n)^2}{n_j (n-j)}$$ and the analysis of variance identity

$V_{jn} = W_n - E_{jn}$ reveals to $$T^2_{jn} = \frac{(n-2) E_{jn}}{(W_n - E_{jn})}$$

The Change Point test is known by comparing h^2_n with maximum of statistics of the allowed j values. If $T^2_{max,n} > h^2_n$, leading to the signal of a change point, then it is in significant matter to compute the maximum likelihood estimators.

$$\hat{\mu}_1 = \frac{S_j}{j,} \quad \hat{\mu}_1 = \frac{(S_n - S_j)}{n-j} \text{ and } \hat{\sigma}^2 = \frac{V_{jn}}{n-2}$$

(the customary variance estimator), using the value 'j' leading to the maximum. The estimators are somewhat biased even though the maximum likelihood estimator are calculated [3,4]. The maximizing j is that which maximizes E_{jn}, so the searching step need only to evaluate E_{jn} for each j, making further T^2_{jn} calculation necessary only for the maximizing E_{jn}. Thus, while at process reading number n there are n-1 calculations to be performed, each involves only about the floating point operations, so even if n were in the tens of thousands calculating $T^2_{max,n}$ would still be a trivial calculation. The ever-growing storage requirement for the two tables might be more inconvenient. It is acceptable to restrict the search for change point to the most recent 'ù' instance, if this can be limited along with the size of the resulting search. It is done only when one must keep a table of only the 'ù' most recent S_j and W_j values. Willsky and Jones [1] discussed the 'window' approach which is different from above method, in that observations more than w time periods into the past are not lost, since they are summarized in the window's leftmost S and W entry. The lost is the ability to split at these old instants. Appropriate values for the table size W may be in the 500 to 2000 range. It is very large as no interesting structure is lost, but small enough to compute for each new reading to less than 20000 operations.

IV. CHANGE POINT METHOD WITH WEIBULL DISTRIBUTION

Consider the model with none of the parameters known; we can test the presence of change point with another general Likelyhood ration test. This test is a two sample t-test between before shift and after shift of the sequence, maximised across all possible change points [12]. For a given change point 'j' where $1 \leq j \leq n-1$, let

$X_{jn} = \sum_{j-1}^{j} X_{ij}/_j$ be the means of the 'j' observations before shift $X^*_{jn} = \sum_{i=j+1}^{n} X_1/_{(n-j)}$

be the mean of the (n-j) observations after shift. The residual sum of squares i.e. V_{jn} is given by

$$V_{jn} = \sum_{i=1}^{j} (X_i - X_i)^2 + \sum_{i=j+1}^{n} (X_i - \overline{X}^*_j)^2 \qquad (2)$$

here we assume that there is a single change point approach at epoch 'j', X_{jn} and X^*_{jn} are the Maximum Likelyhood Estimators of μ_1, μ_2 and $\sigma^2_{jn} = \frac{V_{jn}}{n-2}$ is the usual pooled estimator of σ^2. A conventional two sample t statistic for comparing these two segments could be

$$Tjn = \sqrt{\frac{j\,(n-j)}{n}}\;\frac{\overline{X}_{jn} - \overline{X}^*_{jn}}{\sigma_{jn}} \qquad (3)$$

For a stable process, T_{jn} follows at t – distribution with n-2 degrees of freedom. An out of control signal is obtain as soon as

$$T_{max,n} = \text{Max}\ \{|T_{jn}|, \text{ for } 1 \le j \le n-1\} \qquad (4)$$

exceeds some critical value, h_n.

Hawkins et al. [2] showed that, for any type I error, h_n can be computed by using the Bonferroni inequality. They mentioned that the latter is conservative when a process measurement (n) is large. At same time, they provide empirical control limits, $h_{n,\alpha}$ for different type I error, α, and various number of process observations. Also, for large n values ($n \geq 1$), they proposed the following approximation formula for computing $h_{n,\alpha}$.

$$h_{n,\alpha} = h_{n,\alpha}\left[0.677 + 0.019\ 1n\,(\alpha) + \frac{1\text{-}0.155\ \ 1n(\alpha)}{n-6}\right] \qquad (5)$$

where in (.) is the natural log function.

In the case of general change point formulation in which either or both of the parameters μ, σ may shift at the change point τ. Sullivan and Woodall [10] discuss this formulation and the resultant generalised likely hood ratio test. It provides a single diagnostic to detect shifts in either the mean or the variance or in both. This has the disadvantage of normality assumption. Furthermore, while bounds, approximations and extreme value results are known for the hull distribution of $T_{max,n}$, there is any hardly sample theory for Sullivan and Woodall statistic. Based on these reasons we will consider for the change point formulation to mean the formulation in the third scenario i.e. none of the process parameters is consider known exactly.

In Phase I, with its static set of data $X_{1,}$ X_2,.......X_n, traditional fixed sample statistical methods are appropriate. So, for example, it is appropriate to calculate $T_{max,n}$ for the whole data set and test it against a suitable fractiles of the null distribution of the test statistic for that value of n. If the analysis indicates a lack of control in the Phase I data set, more data will be gathered after process adjustment until a clean data set is achieved.

Phase II data are the process readings gathered subsequently unlike the fixed set of Phase I, they are from a never-ending stream. As each new reading accrues, the SPC check is re-applied. For this purpose, fixed significance level control limits are not appropriate: rather, concern is with the run lengths, both in-and out-of-control. A convenient summary of the frequency of false alarms is the in-control average run length (ARL), which should be large, and self-contained GLR in which the maximised Likelyhood and the Likelyhood ratio are used both detection and estimation.

In traditional methods viz., Shewart, CUSUM, EWMA charts, require a Phase I data set to have parameter estimates that can be used in the Phase II calculations. These methods require on to draw a connectional line below the Phase I data and separate the estimated data [Phase I] from the SPC data [Phase II]. On contrast, in change point formulation one does not assume known parameters and hence does not require the estimates produced by a Phase I. Once the preliminaries are complete and the initial process stability has been achieved, the change point allows to go seamlessly into SPC in which, at each instant, all accumulated process readings are analysed and all data is used to test for the presence of a change point. When process remains in control, it also provides on ongoing stream of every improving estimates of the parameters.

The change point model is a schematic approach in which each new observation X_n is added to the data set, the change point statistic $T_{max,n}$ is calculated for the sequence $X_{1,}$ X_2,.......X_n. If $T_{max,n} > h_n$. Where $\{h_1 n\}$ is a suitably chosen sequence of control limits, then we conclude that there has been a change in mean. The important point is the choice of the control limits sequence $\{h_{1,n}\}$. The ideal would be a sequence $\{h_{1,n}\}$, such that the hazard or alarm rate [the conditional probability of a false alarm at any 'n', given that there was no previous alarm] was a constant α, as in case with the Shewart chart. When the hazard rate is constant the in-control ARL would be $\frac{1}{\alpha}$. This approach was used by Margavio et al. [9] in the context of an EWMA chart in context known parameters the false alarm changes point over time. Margavio et al. [9] derived control limits sequence that would fix the false alarm for EWMA chart to a specified value. The present paper attempts to obtained distribution of $T_{max,n}$.

In fact obtaining distribution of $T_{max,n}$ the sequence is far from being able to provide distributional theory sequence. Hence an attempt is made using simulation to tackle this problem. In fact the change point does not depend on the parameters estimation from Phase I so it is possible to star testing for a change point with the third process reading. Table 1 is obtained by simulation of 10 million sequence of length 200 using weibull distribution. This table shows the control limits for á value of 0.05, 0.02, 0.01, 0.005, 0.002 and 0.001, corresponding in control ARLs of 20, 50, 100, 200, 500 and 1000 for different 'n' values ranging from 3 to 200.

TABLE 1: CUTOFFS $h_{n,\alpha}$ FOR SAMPLE SIZE n AND HAZARD RATE α STARTING AT SAMPLE 3

n	0.05	0.02	0.01	0.005	0.002	0.001
3	25.4517	63.65674	127.3213	254.6466	636.6192	1273.239
4	7.648804	12.18613	17.27718	24.46427	38.71047	54.75856
5	5.391949	7.453319	9.4649	11.98376	16.32633	20.60409
6	4.604095	5.951373	7.173182	8.610302	10.9155	13.03367
7	4.219309	5.247417	6.13837	7.146386	8.692542	10.05301
8	3.997061	4.848107	5.563179	6.351018	7.52302	8.523984
9	3.855226	4.594638	5.202189	5.858792	6.814035	7.612264
10	3.758586	4.421439	4.956973	5.527352	6.343208	7.013689
11	3.689662	4.296806	4.780913	5.290654	6.010132	6.593683
12	3.638848	4.203635	4.649243	5.114154	5.763356	6.284337
13	3.600447	4.131928	4.547656	4.978133	5.573976	6.047961
14	3.570882	4.075467	4.467333	4.870557	5.424579	5.862067
15	3.5478	4.030193	4.40256	4.783685	5.304082	5.712457
16	3.529593	3.993347	4.349476	4.712323	5.205113	5.589751
17	3.515129	3.962993	4.305383	4.65286	5.122586	5.487513
18	3.503588	3.937734	4.268344	4.602711	5.052882	5.401188
19	3.494364	3.916538	4.23693	4.559982	4.993362	5.327467
20	3.487002	3.898627	4.210068	4.52325	4.942055	5.263887
22	3.476538	3.870409	4.166892	4.463679	4.858423	5.160113
24	3.470215	3.849692	4.134158	4.417862	4.793551	5.079398
26	3.466777	3.834327	4.108908	4.381917	4.742126	5.01518
28	3.465401	3.822883	4.089178	4.353269	4.700642	4.963144
30	3.465522	3.81438	4.073619	4.33015	4.666698	4.92034
35	3.470065	3.801886	4.047307	4.289134	4.604778	4.841425

(Table Contd. on next page)

n	0.05	0.02	0.01	0.005	0.002	0.001
40	3.47808	3.797231	4.032486	4.263587	4.564119	4.78858
45	3.487793	3.797093	4.024515	4.247407	4.536458	4.751721
50	3.498296	3.799678	4.02085	4.237231	4.517231	4.725286
60	3.519917	3.809271	4.021035	4.227685	4.494256	4.691687
70	3.541075	3.821638	4.026596	4.226261	4.483276	4.673205
80	3.561185	3.83498	4.03474	4.229106	4.47892	4.663231
90	3.580104	3.848487	4.044119	4.2343	4.478461	4.658384
100	3.59785	3.861779	4.054036	4.24081	4.480391	4.656776
125	3.637614	3.893147	4.07909	4.259533	4.490664	4.660564
150	3.671928	3.921475	4.102962	4.278972	4.504237	4.669669
175	3.701978	3.946974	4.125098	4.297784	4.51868	4.680807
200	3.728651	3.970029	4.145496	4.315569	4.533051	4.692607

It can be noted that from table1, as the 'α' values decreases the control limits $h_{n,\alpha}$ decreases sharply initially, but then stabilizes. Similar type of behaviour can be observed even with Normal distribution [2].It may not be reasonable that start testing at the third observation. However there are cases where the process shift occurs at third observation. In practice the particular should gather a modest number of observations to get an initial verification that the Exponential distribution was a reasonable fit. Then only the formal change point can be applied. In view of this our main simulation is based on assumption of 9 readings without testing, with testing starting at the 10^{th} observation, however 9 readings itself is hard to believe in a Exponential distribution for the Quality variable. Perhaps it is making reasonable compromise between the conflicting desires.

The same 10 million sequence of length 200 are used to find out cut offs points up to n =200. The resulting control limits are presented in table 2, these are our recommendation for implementation of change point SPC schemes.

TABLE 2: CUTOFFS $h_{n,\alpha}$ FOR SAMPLE SIZE n AND HAZARD RATE α STARTING AT SAMPLE 10

n	0.05	0.02	0.01	0.005	0.002	0.001
10	3.502147	4.012155	4.965784	5.554781	6.124578	7.251478
11	3.113349	3.581442	4.446466	4.989267	5.523502	6.559912
12	2.956393	3.387537	4.193278	4.691289	5.173446	6.126179
13	2.844282	3.249034	4.012429	4.478447	4.923406	5.816369
14	2.760199	3.145156	3.876793	4.318816	4.735875	5.584012
15	2.694801	3.064362	3.771298	4.194659	4.590019	5.40329
16	2.642482	2.999727	3.686902	4.095333	4.473333	5.258713
17	2.599676	2.946844	3.617851	4.014066	4.377863	5.140422
18	2.564004	2.902774	3.560308	3.946343	4.298305	5.041846
19	2.53382	2.865485	3.511618	3.88904	4.230987	4.958436
20	2.507949	2.833523	3.469884	3.839923	4.173285	4.886941

(Table Contd. on next page)

n	0.05	0.02	0.01	0.005	0.002	0.001
22	2.465907	2.781584	3.402066	3.760107	4.07952	4.770763
24	2.433208	2.741187	3.349318	3.698028	4.006591	4.680402
26	2.407049	2.708869	3.30712	3.648365	3.948249	4.608113
28	2.385646	2.682428	3.272595	3.607732	3.900514	4.548968
30	2.36781	2.660393	3.243823	3.573871	3.860735	4.49968
35	2.333983	2.618603	3.189257	3.509651	3.785291	4.406203
40	2.310105	2.589104	3.15074	3.46432	3.732037	4.340219
45	2.29235	2.567169	3.122098	3.430612	3.692438	4.291154
50	2.27863	2.55022	3.099967	3.404565	3.661839	4.25324
60	2.258813	2.525737	3.067998	3.366941	3.61764	4.198476
70	2.245188	2.508905	3.04602	3.341075	3.587253	4.160826
80	2.235246	2.496622	3.029982	3.3222	3.565079	4.133351
90	2.227671	2.487263	3.017763	3.307819	3.548184	4.112418
100	2.221707	2.479896	3.008143	3.296497	3.534884	4.095939
125	2.211184	2.466895	2.991167	3.276519	3.511414	4.066858
150	2.204314	2.458408	2.980086	3.263477	3.496093	4.047874
175	2.199477	2.452433	2.972283	3.254294	3.485304	4.034507
200	2.195887	2.447997	2.966491	3.247477	3.477297	4.024585

V. PERFOMENCE OF THE CHANGE POINT APPROACH WITH WEIBULL DISTRIBUTION

In this section, we are interested to demonstrate the change point model behaviour when we compare CUSUM chart performance in the contest of Weibull distribution. Here also we have developed 10 million sequence of length 200 with varying á value of 0.05, 0.02, 0.01, 0.005, 0.002 and 0.001, corresponding in control ARLs of 20, 50, 100, 200, 500 and 1000 for different ‘n’ values ranging from 3 to 200. These values are depicted in table 3. In the context of Weibull distribution also (i.e. when the phenomenon follows Weibull distribution) for each α, control limits $h_{n,\alpha}$ describes sharply initially but then stabilize

The above points are illustrated in table 3 by considering α value 0.02, 0.01, 0.005 and 0.002. A shift of size $\delta \in \{0, 0.5, 0.6, 0.75, 1.25, 1.5, 1.75, 2, 2.25, 2.5, 3\}$ was introduced $\tau \in \{10, 25, 50, 100, 250\}$. The values presented in the Table 3 are the ARL’s of the change point procedures. These were calculated by simulating a data series, adding the shifts to all X_i’s, $i > \tau$ and counting the number of readings from the occurrence of the shift until the chart is signalled. Any sequence which is signalled before time τ was discovered the appropriate formula h_n is used. So that the in control ARL’s differs from normal.

It is clear from the table that the ARL’s are affected by the amount of history is gathered before the shift, with a faster response carrying with more history. These ARL’s are also depends α and δ. It can also be observed that α is large or δ is large the ARL tends to be smaller, as one would anticipating. For different δ values in the range 0 to 3, we present resulting ARLs in Fig. 1 to 4.

From the figures we note that the ARLs are taken on a log scale for clear comparison. In this context the change point slightly worse than k=0.25 CUSUM chart for small shifts in the process average, for medium shifts also the change point model inferior to k=0.5 and k=1 CUSUM chart. However there is a marginal variation between change point model and CUSUM with k=2.5. For large shifts also there is no much of difference between change point model and CUSUM chars. The behaviour of change point model when the phenomenon follows Weibull distribution is similar to the behaviour of change point model with CUSUM chart in the context of normal distribution [2].

TABLE 3: THE ARL OF THE CHANGE POINT PROCEDURE WHEN SHIFT OCCURS AT THE 'START' POSITION WITH SIZE 'α'

		α							
α	**start**	**0**	**0.25**	**0.5**	**1**	**1.5**	**2**	**2.5**	**3**
0.02	**10**	210.7262	14.56642	7.543947	3.840722	2.576132	1.938022	1.553275	1.295988
	25	84.29046	5.826568	3.017579	1.536289	1.030453	0.775209	0.62131	0.518395
	50	42.14523	2.913284	1.508789	0.768144	0.515226	0.387604	0.310655	0.259198
	100	21.07262	1.456642	0.754395	0.384072	0.257613	0.193802	0.155327	0.129599
	250	8.429046	0.582657	0.301758	0.153629	0.103045	0.077521	0.062131	0.05184
0.01	**10**	248.0634	17.14735	8.880612	4.521236	3.032581	2.281409	1.82849	1.525616
	25	99.22537	6.858941	3.552245	1.808494	1.213032	0.912563	0.731396	0.610246
	50	49.61268	3.429471	1.776122	0.904247	0.606516	0.456282	0.365698	0.305123
	100	24.80634	1.714735	0.888061	0.452124	0.303258	0.228141	0.182849	0.152562
	250	9.922537	0.685894	0.355224	0.180849	0.121303	0.091256	0.07314	0.061025
0.005	**10**	285.4007	19.72829	10.21728	5.20175	3.48903	2.624795	2.103705	1.755244
	25	114.1603	7.891315	4.086911	2.0807	1.395612	1.049918	0.841482	0.702098
	50	57.08014	3.945657	2.043456	1.04035	0.697806	0.524959	0.420741	0.351049
	100	28.54007	1.972829	1.021728	0.520175	0.348903	0.262479	0.210371	0.175524
	250	11.41603	0.789131	0.408691	0.20807	0.139561	0.104992	0.084148	0.07021
0.002	**10**	334.7579	23.1401	11.98425	6.10134	4.092423	3.078727	2.46752	2.058796
	25	133.9031	9.256039	4.793701	2.440536	1.636969	1.231491	0.987008	0.823518
	50	66.95157	4.628019	2.396851	1.220268	0.818485	0.615745	0.493504	0.411759
	100	33.47579	2.31401	1.198425	0.610134	0.409242	0.307873	0.246752	0.20588
	250	13.39031	0.925604	0.47937	0.244054	0.163697	0.123149	0.098701	0.082352

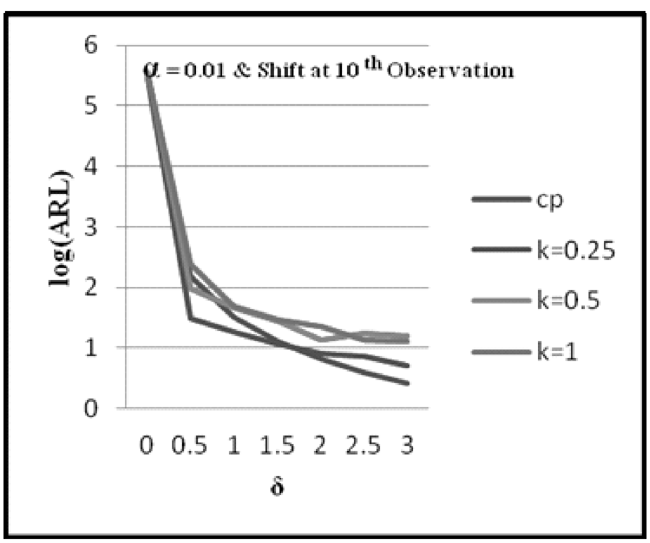

Fig. 1

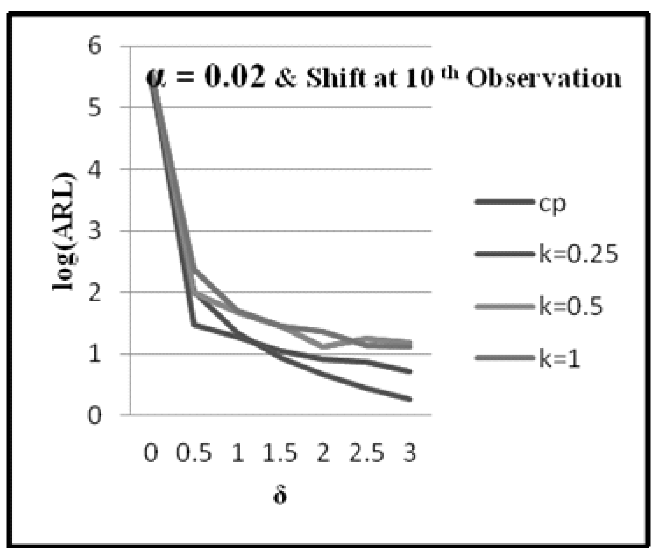

Fig. 2

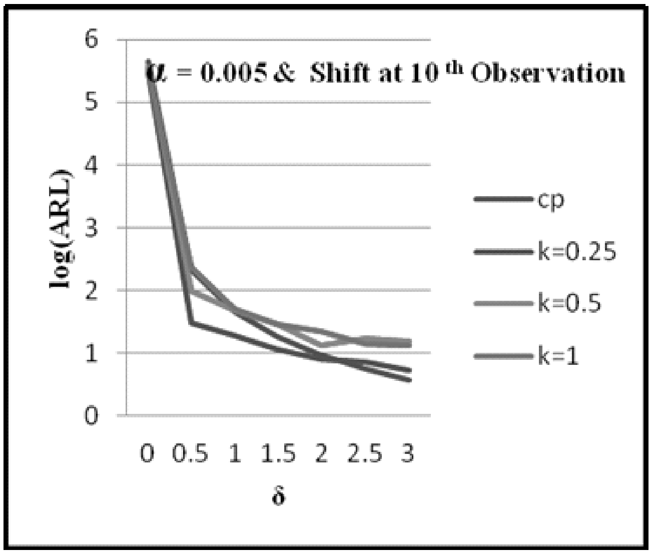

Fig. 3

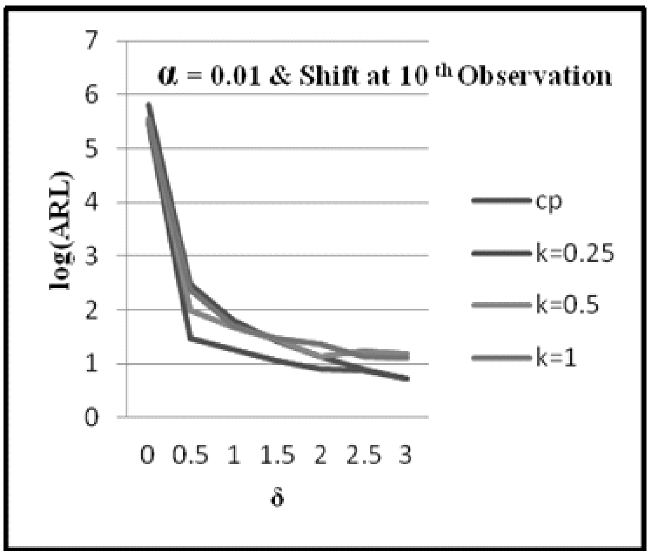

Fig. 4

VI. CONCLUSIONS

In the present paper control limits or cut off points [$h_{n,\alpha}$] are obtained by simulation of 10 million sequence of length 200 with the process readings start at 3 and 10 are obtained in the context of Weibull distribution. i.e. when the phenomenon under considerations follows Weibull distribution. This paper also provides a comparative study between change point model and CUSUM chart by considering ARLs in the context of Weibull distribution. We found that the performance of change point model is superior to CUSUM chart in the context of Weibull distribution, it is found that the change point model is slightly worse than CUSUM chart in detecting small shifts, which is similar case with the normal distribution [2]. It would be very interesting to study the behaviour of change point model and comparing its performance with CUSUM chart in the context of non normal distributions i.e. when the process average follows non normal distribution

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