Analysis of Limit Cycle Stability and Control for Nonlinear Systems

Ashish Kumar Shakya
(M.Tech.),
Electrical Engineering Department,
NIT Kurukshetra, Haryana
ashish1342@gmail.com

Lillie Dewan
Professor
Electrical Engineering Department,
NIT Kurukshetra, Haryana
l_dewanin@yahoo.com

Abstract: *Describing function method is used for stability investigation and prediction of limit cycle in nonlinear systems. If there are uncertainties in the system then the frequency and amplitude of the limit cycle will change accordingly and describing function method can deal with these uncertainties. In this paper attempt has been made to study the stability and robustness of nonlinear systems in the presence of uncertainties in different forms. Barkhausen characteristic equation is used in this paper to determine the robustness of the limit cycle. A robust controller to control the amplitude and the frequency of limit cycle against the parametric perturbations is also explained in this paper.*

Keywords: *Lur'e Problem, Describing Function Method, Limit Cycle, Barkhausen characteristic polynomial, Robust limit cycle controller.*

I. INTRODUCTION

There are several well-developed techniques for analyzing nonlinear feedback systems like Phase plane method, Lyapunov stability, Singular perturbation, Popov, Small-gain theorem etc. To analyze the nonlinear system stability it is important to analyze the limit cycle behavior which can be done with the help of Describing Function (DF) method [1-7]. This method is applicable to systems with separable nonlinearities [8, 9, 10]. The Nyquist plot of the linear subsystem and the negative inverse of the describing function of the linearzed element are drawn in the complex plane. The intersection of these plots is used in an approximate, but useful limit cycle analysis. Using the describing function method it is also possible to analyze the limit cycle behavior of a nonlinear system in the presence of uncertainty [11]. This paper explores the problem of stability analysis of nonlinear systems (nonlinearity in feedback and feed-forward position) in the presence of uncertainty. Lur'e control system problem has a forward path that is linear timeinvariant, and a feedback path that contains a memory-less, possibly time-varying, static nonlinearity [5]. The nonlinear Lur'e problem can be linearized by describing function method (DF). A formula based on Loeb criterion determine whether the limit cycle is locally stable or not[12]. In this paper unstructured uncertainty has been modeled in additive and multiplicative form in the system to analyze the nonlinear systems stability. In addition of stability analysis of nonlinear systems a robust controller is also designed in this paper which can control the amplitude and frequency of limit cycle in the presence of uncertainties [14, 15].

The rest of paper is organized as follows- Brief introduction of describing function method used for nonlinear system analysis is given in section II. A method of limit cycle stability analysis is discussed in section III. Section IV presents additive and multiplicative uncertainty modeling. Analysis of Robust stability of limit cycle is dealt in section V. An example has been given in section VI for the explanation of sections explained above. A controller to control the amplitude and the frequency of limit cycle is also explained in section VII with an example. Finally section VIII concludes the paper.

II. DESCRIBING FUNCTION METHOD OF THE NONLINEAR SYSTEM ANALYSIS

The describing function method is used to determine the limit cycle and dynamical behavior of the nonlinear systems [4-7]. Consider a feedback system shown in Fig.1 where G(s) represents a linear element, while N represents a nonlinear element existing in the feedback path.

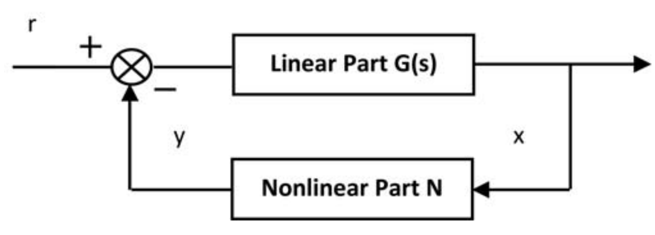

Fig. 1. Lur'e System

If the nonlinearity assumed to be an odd function and the transfer function G (s) has the low pass characteristics then the nonlinearity can be replaced by the describing function N (X) which is the complex ratio of the fundamental component of the output y (t) and the sinusoidal input x (t), that is:

$$N(X, w) = \frac{Y_1 e^{j\varphi 1}}{x} \qquad \ldots(1)$$

Where $Y_1 = \sqrt{A_1^2 + B_1^2}$ and $\varphi_1 = \text{arctg} \frac{A_1}{B_1}$

Where A_1 and B_1 are Fourier coefficients. Now the overall system can be treated as a linear system and the characteristic equation can be written as:

$$1 + N(X,w)\, G(jw) = 0 \qquad \ldots (2)$$

III. STABILITY ANALYSIS OF LIMIT CYCLE

If the limit cycle is stable, the states of the system will return to it after perturbations. Any small perturbation from the closed trajectory would cause the system to return to the limit-cycle making the system stick to the limit-cycle. So the limit cycle and nonlinear system stability has a close relationship. Consider the nonlinear Lur'e system as seen in Fig.1 and the nonlinear element N is even symmetry and memory-less, the DF(describing function) of the nonlinear element N is denoted as N(X). Assume that there is an intersection X_0, w_0 of Nyquist plot of Eq.(2),

$$\frac{\partial N(X, w)}{\partial X}\bigg|\, X_0, w_0 \neq 0, \; \frac{\partial}{\partial_w} \text{ Im}G(jw)\bigg|\, w_0 \neq 0$$

And all the roots of $1 + N(X_0)G(s) = 0$ are Hurwitz except for $\pm\, jw_0$

$$\text{if} \qquad N'(X) \;\text{Re}\,[G'(jwo)] < 0 \qquad \ldots(3)$$

then there exists a locally stable limit cycle.

On the other hand,

$$\text{if} \qquad N'(X)\; \text{Re}\,[G'(jwo)] > 0 \qquad \ldots (4)$$

then there exist an unstable limit cycle[12].

IV. UNCERTAINTY MODELING

Model uncertainty can be divided into two general categories: structured and unstructured uncertainty [13]. Structured uncertainty assumes that the uncertainty is modeled and we have ranges and bounds for uncertain parameters in the system. Unstructured uncertainty assumes less knowledge of the system. We only assume that the frequency response of the system lies between two bounds. Unstructured uncertainty can be modeled in different ways: additive and multiplicative uncertainty.

(1) *Additive uncertainty*: suppose we model a system by G(s) and actual system is given by $\tilde{G}$ (s)

$$\tilde{G}(s) = G(s) + \Delta_a(s) \qquad \ldots (5)$$

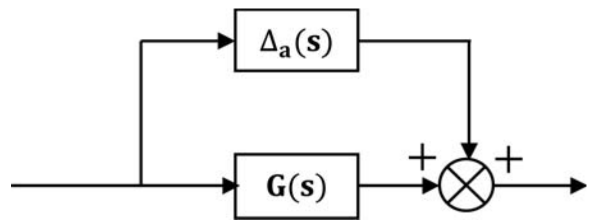

Fig. 2. Additive perturbation configuration

Therefore, the model error, or the additive uncertainty, is given by

$$\Delta_a(s) = \tilde{G}(s) - G(s) \qquad \ldots(6)$$

(2) *Multiplicative uncertainty*: In the multiplicative uncertainty case, we assume the true model, $\tilde{G}(s)$ is given by

$$\tilde{G}(s) = [1 + \Delta_m(s)]\, G(s) \qquad \ldots(7)$$

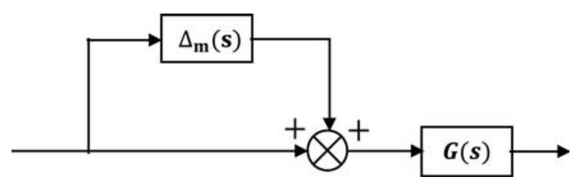

Fig. 3. Multiplicative perturbation configuration

The uncertainty, or the model error, is given by

$$\Delta_m(s) = \frac{\tilde{G}(s) - G(s)}{G(s)} \qquad (8)$$

V. ROBUST STABILITY OF LIMIT CYCLE

With the help of Barkhauen characteristic equation robustness of the limit cycle can be determine [12]. This Barkhauen characteristic equation is generated with the help of conventional Barkhauen principle. By applying Barkhauen principle in the feedback system shown in Fig.1 we have

(the phase of $G(j\omega_o)$ is 180 degree),

$$\text{Img } G(jw_d) = 0 \quad \ldots(9)$$

and

$$N(X)_{max} \cdot \text{Re}G(jw_o) > 1 \quad \ldots(10)$$

Where $N(X)_{max}$ is the maximum value of describing function N(X).

Eq. (9) can be written as

$$\text{Im } G(jw_o) = G(s) - G(-s) \Big|_{s=jw_o} = 0 \quad \ldots(11)$$

Eq. (11) can be written as

$$1\text{m } G(jw_o) = \frac{A(s)\,B(-s) - A(-s)\,B(s)}{A(s)\,B(-s)} \Bigg|_{s=jw_o} = 0 \quad \ldots(12)$$

Where A(s) and B(s) are the numerator and denominator of the transfer function G(s) respectively. From Eq. (12)

$$C(s) = A(s)\,B(-s) - A(-s)\,B(s) \Big|_{s=jw_o} = 0 \quad \ldots(13)$$

Eq. (13) is called Barkhausen characteristic polynomial. The robustness of nonlinear system is directly dependent on the value of $|C'(jw_o)|$.

VI. Example (1)

Consider a Lur'e problem as seen in Fig. 1, which has the plant transfer function G(s) is such as

$$G(s) = \frac{0.2412s^3 + 0.60444s^2 + 0.03846s + 0.0615}{s^5 + 5.06s^4 + 0.4085s^3 + 0.5436s + 0.0063s + 0.004} \quad \ldots(14)$$

and the nonlinearity which is a combination of dead zone, coulomb & Viscous friction and saturation nonlinearity. The describing function of this nonlinearity will be

$$N(X) = \frac{1}{\pi x}\left[2\left(\sin^{-1}\frac{1}{x} - \sin^{-1}\frac{0.5}{x}\right) + \frac{2}{x}\sqrt{1-\left(\frac{1}{x}\right)^2} + \frac{1}{x}\sqrt{1-\left(\frac{0.5}{x}\right)^2}\right] \quad \ldots(15)$$

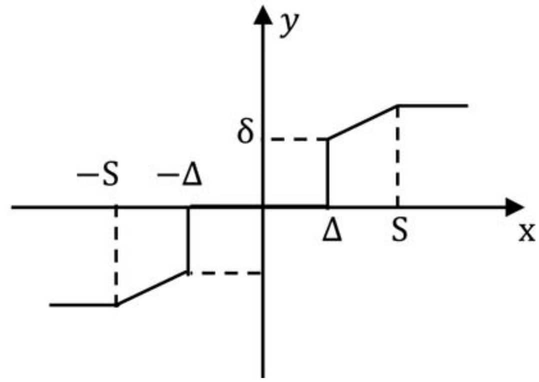

Fig. 4. Sinusoidal response of nonlinearity

When the values of saturation S=1, dead zone Δ=0.5 and Coulomb friction value 0.5 with coefficient of viscous friction is δ=1.

The point of intersection of G(jw) and -1/N(X) can be obtained by describing function method and which is $\omega_o = 0.278$ rad/sec, $X_o = 2.31$. With Eq.(3), it can be founded that the limit cycle is locally stable. Fig.5 shows the simulation model.

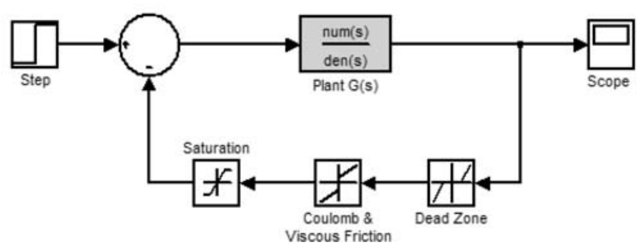

Fig. 5. Simulation model

From Eq.(13), we have the Barkhausen Characteristic Polynomial

C(s)= 0.4824*s^7 +0.1007*s^5 +0.0079*s^3 + 0.0002*s ...(16)

The roots of Eq. (16) are -0.0789 $\pm$0.2948j, 0.0798 $\pm$ 0.2948j, $\pm$0.2183j, 0

$|C'(0.278j)|$ = 1.8306e - 004, it can be observed that the limit cycle is not robust. Now let's consider the following five parameter's perturbations.

Perturbation (1): $-0.04s^2 - 0.002s - 0.004$ numerator of G(s).

Perturbation (2): 0.006s + 0.001 in the numerator of G(S)

Perturbation (3): $0.1s^3 - 0.002s^2 + 0.003s + 0.01$ in the denominator of G(s)

Perturbation (4): $0.2s^4 + 0.01s^3 - 0.025s^2 - 0.001s$ in the denominator of G(s).

Perturbation (5): $\frac{0.2s + 0.1105}{0.15s+0.1}$

un-modeled dynamics of G(s)

Now determine the change in frequency and amplitude of limit cycle when we are dealing with plant transfer function given in Eq.(14), with given parameter's perturbations. The percentage change of amplitude and frequency when the perturbations are in additive form can be seen in Table 1. Fig. 6 shows the simulation results.

It can be seen that with perturbation (3) and (5), the limit cycle disappear when the perturbations are in additive form and for perturbation (1) the percentage of amplitude's change is - 11.68% and the percentage of frequency's change is +6.11%. With perturbation (2), the percentage of amplitude's change is +6.06% and the percentage of frequency's change is +5.03%.

With perturbation (4), the percentage of amplitude's change is - 6.06% and the percentage of frequency's change is -1.44%.

The percentage change of amplitude and frequency when the perturbations are in multiplicative form can be seen in Table 2 and the Fig.7 shows the simulation results.

Table 1

Perturbation	Amplitude	Frequency	% change in Amplitude	% change in Frequency
Nominal Plant	2.31	0.278	–	–
(1)	2.04	0.295	–11.68	+6.11
(2)	2.45	0.292	+6.06	+5.03
(3)	Limit cycle disappear (unstable behavior)			
(4)	2.17	0.274	-6.06	-1.44
(5)	Limit cycle disappear (unstable behavior)			

Table 2

Perturbation	Amplitude	Frequency	% change in Amplitude	% change in Frequency
Nominal Plant	2.31	0.278	–	–
(1)	2.37	0.274	+2.59	-1.44
(2)	2.31	0.278	0	0
(3)	Limit cycle disappear (unstable behavior)			
(4)	Limit cycle disappear (unstable behavior)			
(5)	2.19	0.288	-5.19	+3.59

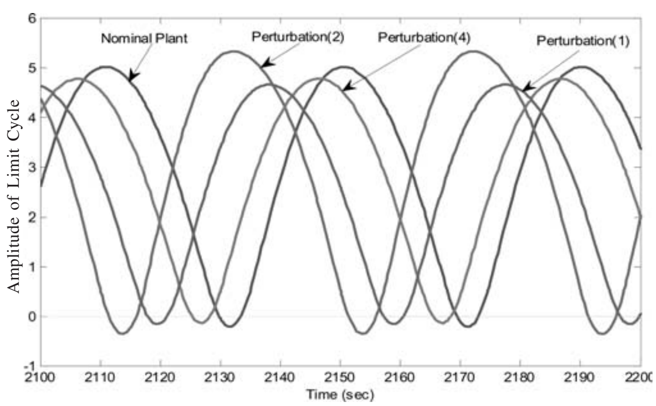

Fig. 6. Limit cycles for the system given in Eq.14 with additive perturbation

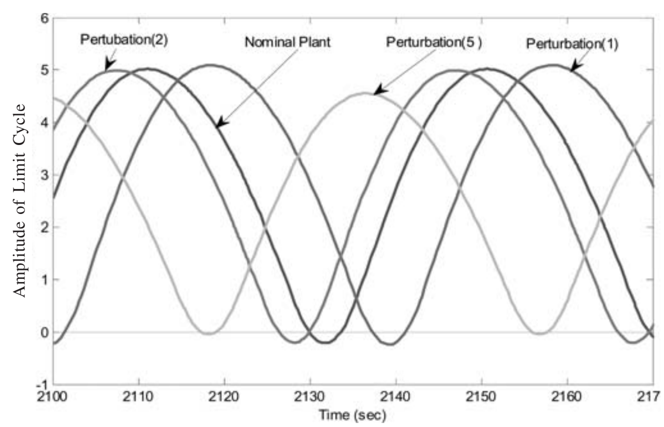

Fig. 7. Limit cycles for the system given in Eq.14 with multiplicative perturbation

It can be seen that with perturbation (3), (4) the limit cycle disappear when the perturbations are in multiplicative form and for perturbation (1), the percentage of amplitude's change is +2.59% and the percentage of frequency's change is -1.44%. With perturbation (2), the percentage of amplitude's change is 0% and the percentage of frequency's change is 0%. As for perturbation (5), the percentage of amplitude's change is -5.19% and the percentage of frequency's change is +3.59%.

VII. ROBUST LIMIT CYCLE AMPLITUDE AND FREQUENCY CONTROLLER

In the previous sections the methods to analyze the stability and robustness of nonlinear Lur'e system is given. Now in this section a design procedure of a robust controller is given which can control the characteristics of limit cycle in the presence of uncertainties [14,15].

This design procedure is based on describing function method. In this design procedure the linear part of the feedback loop is designed such that the intersection point of graphs of linear and negative inverse of nonlinear part is shifted to get the desired limit cycle characteristics.

Suppose the desired limit cycle characteristics are (Xd, wd) and the intersection point corresponding to the desired limit cycle characteristics is (a,0). (Imaginary part is taken zero in this discussion)

According to describing function method:

Desired linear part of feedback loop

$$H(s) = -\frac{1}{N(X,\ w_d)} = 1 + j0 \qquad \text{where } a < 0$$

H(s) is the desired transfer function of feedback loop and

$$H(s) = C(s)\ G(s) \qquad \ldots(17)$$

So after calculating the H(s) the controller can be computed as

$$C(s) = H(s)G^{-1}(s) \qquad \ldots(18)$$

If the transfer function H(jw) intersect $-\frac{1}{N(X_d, w_d)}$ perpendicularly at point (a,0) then the limit cycle will we robust against the uncertainties.

Some changes have to be done in C(s) after designing due to some implementation issues and desired characteristics.

Design Procedure:

Suppose the form of H(s) is

$$H(s) = \frac{n_u s^u + n_{u-1} s^{u-1} + \ldots + n_1 s + n_0}{s^v + d_{v-1} s^{v-1} + \ldots + d_1 s + d_0} \qquad \ldots(19)$$

Where the $n_u, \cdots n_o$ are numerator coefficients and $d_u, \cdots d_o$ are denominator coefficients which have to be designed according to the requirements.

Assume a simple form of H(s) for further calculation

$$H(s) = \frac{n_1\ s + n_0}{s^3 + d_2 s^2 + d_1 s + d_0} \qquad \ldots(20)$$

The requirements according to which the coefficients of H(s) should be chosen are:

(I) Intersection point:

$$H(jw_d) = a = -\frac{1}{N(X_d,\ w_d)} \qquad \ldots(21)$$

From Eq. (20) and (21)

$$\frac{jw_d n_1 + n_0}{(jw_d)^3 + d_2 (jw_d)^2 + d_1 (jw_d) + d_0} = a \qquad \ldots(22)$$

By solving Eq. (22)

$$n_0 = ax \qquad \ldots(23)$$

$$n_1 = ay \qquad \ldots(24)$$

where

$$x = d_0 - d_2\ w_d^2 \qquad \ldots(25)$$

$$y = d_1 - w_d^2 \qquad \ldots(26)$$

One of the requirements for describing function method is the transfer function H(s) of linear part has the low pass characteristics i.e. $d_0 = 0$.

2. Stability Condition:

The characteristics eq. for the system given in Fig. 8 will be

$$1 + H(s)\ N(X,w) = 0 \qquad \ldots(27)$$

(for nominal limit cycle)

Consider $N(X,w) = K = -\frac{1}{a}$ ...(28)

Now with the help of Eq. (20), (27), (28) the characteristics Eq. can be written as

$$s^3 + d_2 s^2 + (d_1 + Kn_1) s + Kn_0 = 0 \quad ...(29)$$

Consider $N(X,w) = K = -\frac{1}{a} + \Delta$ ...(30)

(for perturbed limit cycle and Δ is close to zero)

Now using Eq. (29) and (30) characteristics Eq. for perturbed system will be

$$s^3 + d_2 s^2 + \left(d_1 - \frac{n_1}{a} + \Delta n_1 \right) s - \frac{n_0}{a} + Dn0 = 0 \quad ...(31)$$

Now applying Routh-Hurwitz criterion in Eq. (31) these conditions for stability can be obtained:

$$d_2 > 0 \quad ...(32)$$

$$n_2 > 0 \quad ...(33)$$

(3) Robustness:

For the robustness of limit cycle the H(jw) and -1/N(X) should intersect perpendicularly. Thus the robustness condition for limit cycle against the uncertainties is

$$\left[\frac{dR\ [H(jw)]}{dw} \right]_{w_d} = 0 \quad ...(34)$$

From Eq. (20), (23), (24), (34) and with $d_o = 0$

$$-d_2^2 + y = 0 \quad ...(35)$$

According to these requirements the coefficient of H(s) given in Eq. (20) can be calculated and with the help of H(s) controller C(s) given in Eq. (18) can be calculated.

Example (2)

Consider a system given by Fig. 8 where G(s) and nonlinearity are same as in Example (1).

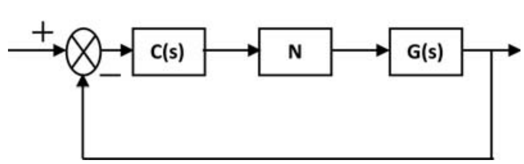

Fig.8. Nonlinear system with nonlinearity in feed-forward path

Now let's consider the following four parameter's perturbations.

Perturbation (1): $-0.06s^2 - 0.0002s - 0.004$ numerator of G(s)

Perturbation (2): 0.006s +0.001 in the numerator of G(s)

Perturbation (3): $0.02s^2 + 0.009s$ in the denominator of G(s)

Perturbation (4): $0.002s^3 - 0.001s^2 - 0.0001s$ in the denominator of G(s)

Now the values of amplitude and frequency of limit cycle with nominal plant are $A_0 = 2.31$, $w_0 = 0.278$ rad/sec and the intersection point of G(s) and -1/N(X) is (-1.9,0). The percentage change of amplitude and frequency when the perturbations are in additive form can be seen in Table 3 and Fig.9 shows the simulation results.

Table 3

Perturbation	Amplitude	Frequency	% change in Amplitude	% change in Frequency
Nominal Plant	2.31	0.278	–	–
(1)	2.52	0.267	+9.09	-3.95
(2)	2.43	0.295	+5.19	+6.11
(3)	2.62	0.327	+13.4	+17.6
(4)	2.74	0.256	+18.6	-7.91

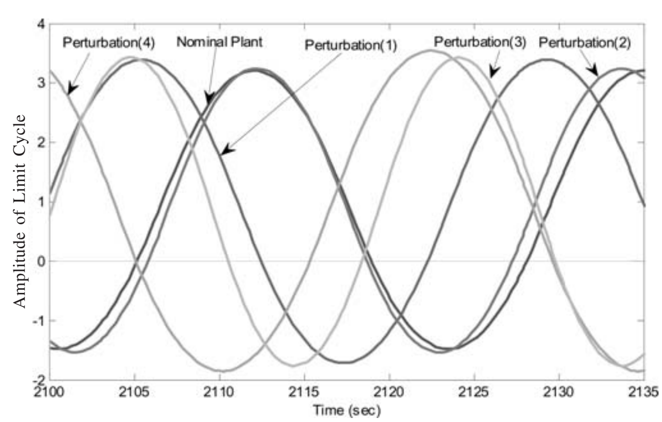

Fig. 9. Limit cycles for the system given in Eq.14 with additive perturbation given in Ex.2

The percentage change of amplitude and frequency when the perturbations are in multiplicative form can be seen in Table 4 and Fig.10 shows the simulation results.

Table 4

Perturbation	Amplitude	Frequency	% change in Amplitude	% change in Frequency
Nominal Plant	2.31	0.278	–	–
(1)	2.38	0.274	+3.03	-1.43
(2)	2.31	0.278	0	0
(3)	2.52	0.266	+9.09	-4.32
(4)	Limit cycle disappear (unstable behavior)			

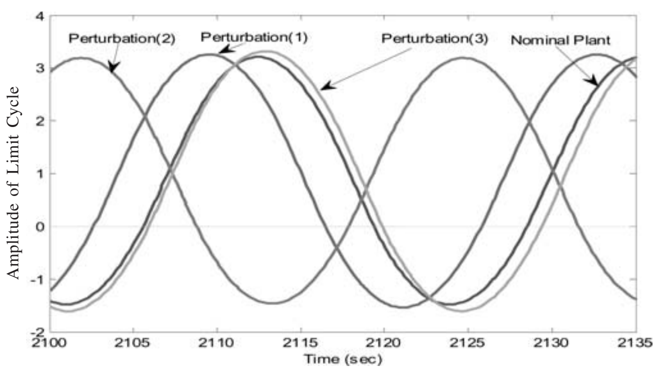

Fig. 10. Limit cycles for the system given in Eq.14 with multiplicative perturbation given in Ex.2

Now design a robust controller to get the desired limit cycle characteristics which is

$X_d = 1.5$, $w_d = 0.5$ rad/sec. According to the desired characteristics the intersection point of H(s) and -1/N(X) should be (-1.3078, 0).

After choosing the value of $d_2 = 1$ with the help of Eq. (23), (24), (25), (26) and (35) the H(s) can be calculated which is

$$H(s) = \frac{-1.3078s + 0.32695}{s^3 + s^2 + 1.25s} \qquad ...(36)$$

Now using Eq. (18)

$$c(s) = \frac{[-1.3078s^6 - 6.2905s^5 + 1.1201s^4 - 0.5774s^3 \; 0.1695s^2 - 0.0032s + 0.0013]}{[0.0241s^6 + 0.6286s^5 + 0.6731s^4 + 0.8555s^3 + 0.1096s^2 + 0.0769s0]}$$

To obtain the desired result a gain of 6 is also added with C(s). Fig.11 shows the simulation model. The Fig.12 and Fig.13 shows the simulation results with additive and multiplicative perturbations after applying the controller.

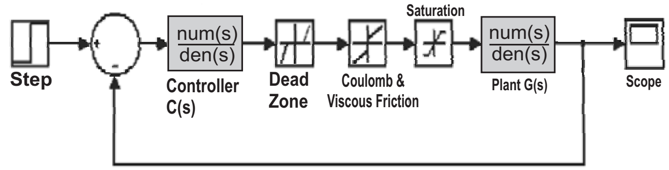

Fig. 11. Simulation model

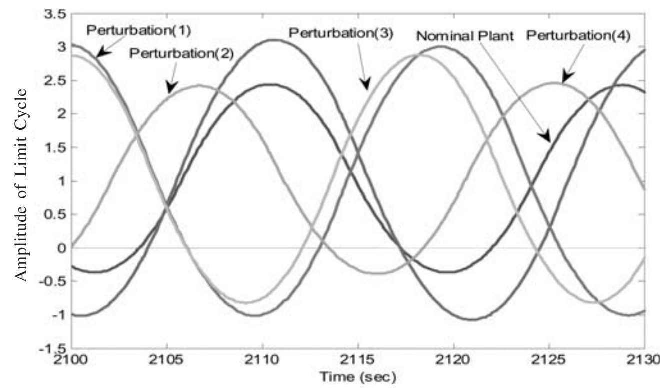

Fig. 12. Limit cycles for the system given in Eq.14 with additive perturbation given in Ex.2

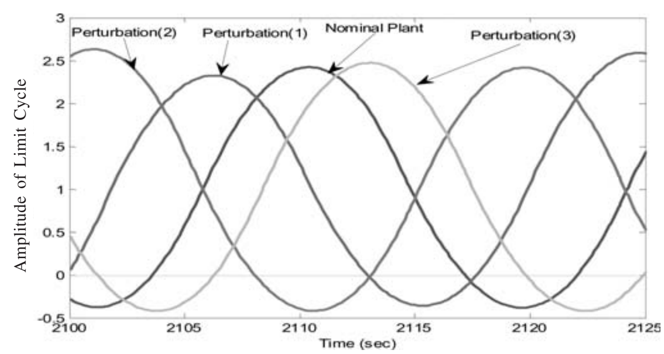

Fig. 13. Limit cycles for the system given in Eq.14 with multiplicative perturbation given in Ex.2

VIII. CONLUSION

In this paper attempt has been made to study the stability and robustness of nonlinear systems in the presence of uncertainty. Through describing function method (DF), behavior of nonlinear system can be analyzed when the nonlinearities are in different form. A formula is given in this paper to determine the stability of limit cycle. A designing procedure of a robust limit cycle controller is also given in this paper which is based on describing function method. According to the requirements loop shaping is done to get the desired limit cycle characteristics. With the help of designed loop transfer function the controller can be determined which is robust against the parametric uncertainties. The simulation results show that the controller designed in this paper can be applied into nonlinear system successfully.

References

(1) Gelb, A., Velde, W.E.V.: Multiple-Input Describing Functions and Nonlinear System Design. McGraw-Hill, New York (1968)

(2) (Steven) Chingyei Chung and J.L. Lin, "A General Class of Sliding Surface for The Sliding Mode Control," IEEE Transactions on Automatic Control, 43, 1998, pp.1509-1512.

(3) (Steven) Chingyei Chung and J.L. Lin, "A transformed lur'e problem for the sliding mode control and chattering reduction," IEEE Trans. on Automatic Control, 1999, 44, (3), 1999.

(4) J.E. Slotine, and Wei-ping Li, Applied nonlinear control, Prentice-Hall, 1991 pp.159.

(5) M. Vidyasagar, Nonlinear system analysis, Prentice-Hall, 1994.

(6) P.A. Cook, Nonlinear dynamical systems, Prentice Hall,1994.

(7) Hassan K. Khalil, Nonlinear systems, Prentice Hall, 1996.

(8) Cook, P., Nonlinear Dynamical Systems. Prentice-Hall, Englewood Cliffs (1986).

(9) Impram, S.T., Munro, N., Describing function analysis of nonlinear systems with parametric uncertainties. In: Proc. UKACC International Conference on Control'98, Swansea, UK, vol. 1, pp. 112–116 (1998).

(10) Tierno, J.E., Describing function analysis in the presence of uncertainty. J. Guid. Control Dyn. 20(5), 956–961 (1997).

(11) Fadali, M.S., Chachavalvoong, N., Describing function analysis of uncertain nonlinear systems using the Kharitonov approach. In: Proc. American Control Conference, Seattle, WA, vol. 4, pp. 2908–2912 (1995).

(12) Chingyei Chung, Shou-Yen Chao, M.F.Lu and S.C. Lee, "Behavior of limit cycle for nonlinear lur'e systems," Proceedings of 2011 8th Asian Control Conference (ASCC) Kaohsiung, Taiwan, May 15-18, 2011.

(13) Design of Feedback control systems Fourth edition Stefani Shahian Savant Hosteller OXFORD.

(14) Neusa Oliveira, Karl Kienitz and Eduardo Misawa "A Describing Function Approach to Limit Cycle Controller Design, " American Control Conference, Minneapolis, Minnesota, USA, June 14-16, 2006.

(15) N.M.F. Oliveira, K.H. Kienitz, E.A. Misawa, "A describing function approach to the design of robust limit-cycle controllers, " Springer Science+Business Media B.V. 2011.

❐