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Stability Criteria of Kapur's Entropy under the Deformed Specification of Exponential Distribution

Abstract: In this paper, we will discuss the stability criteria of generalized entropies of Shannon as Kapur's entropy. This entropy is non-extensive in nature and satisfy the stability condition under the given deformed specifications of the q-exponential distribution and hence gives more convergent experimental observations than other non-extensive entropies .

Keywords— Kapur's entropy, Renyi's entropy, Normalised Tsalli's entropy, Tsalli's entropy.

I. INTRODUCTION

Shannon[7] discussed the concept of generalised entropies. Tsallis entropy [8]-[9] and Kapur's Entropy [10] plays an important role in elaborating non extensive systems. Tsallis [8] introduced non-extensive systems 20 years ago. Lesche [6], showed instability of the Renyi's entropy. Abe [1]-[3], Bashkirov[4] and Cao[5] showed the instability of Renyi's and Normalized Tsallis entropies and stability of Tsallis entropy for thermodynamic system under small deformation of a distribution. Therefore, normalized Tsalli's and Renyi's entropy cannot represent experimentally observable quantities for the q-exponential distributions. Here, it is shown that Kapur's entropy is stable under the given deformed specification of the q-exponential distributions and therefore used for further experimental purposes.

Kapur's Entropy of order and type [10] is defined as:

$$H^{K}_{\alpha,\beta}(p) = \frac{1}{\beta-\alpha} \log \left(\frac{\sum_{i=1}^{N} p_i^{\alpha}}{\sum_{i=1}^{N} p_i^{\beta}} \right), \alpha \neq \beta,\ \alpha, \beta > 0 \quad (1.1)$$

II. STABILITY CRITERIA OF KAPUR's ENTROPY

The idea of stability was given by Lesche [6] while studying Renyi entropy. Let us discuss the concept of stability. Consider that be the channel capacity of the system. Mathematically, it can be expressed as:

$$\|p - p'\| \leq \delta \Rightarrow \left| \frac{C(p) - C(p')}{C_{max}} \right| < \varepsilon \left(\forall \varepsilon > 0, \exists \varepsilon > 0\right) \quad (2.1)$$

for arbitrarily large values of N. where,

$$\|A\| = \sum_{i=1}^{N} |A_i|$$

$\wp_{max} = C(p)$

Consider the following deformed specification of the distribution as given by Abe [1] as:

(i) For $0 < \alpha < 1$:

$$p_i = \delta_{i1},\ p' = \left(1 \frac{\delta}{2} \frac{N}{N-1}\right) p_i + \frac{\delta}{2} \frac{1}{N-1} \quad (2.2)$$

under the condition, $\sum_{i=1}^{N} (p_i)\alpha = 1$ (2.3)

$$\sum_{i=1}^{N} (p'_l)^{\alpha} = \left(1 - \frac{\delta}{2}\right)^{\alpha} + \left(\frac{\delta}{2}\right)^{\alpha} (N-1)^{1-\alpha} \quad (2.4)$$

(ii) For $\alpha < 1$:

$$p_i = \frac{1}{N-1}(1 - \delta_{i1}),\ p'_i = \left(1 - \frac{\delta}{2}\right) p_i + \frac{d}{2}\delta_{i1} \quad (2.5)$$

under the condition, $\sum_{i=1}^{N} (p_l)^{\alpha} = (N-1)^{1-\alpha}$ (2.6)

$$\sum_{i=1}^{N} (p'_l) \left(\frac{\delta}{2}\right)^{\alpha} + \left(1 - \frac{\delta}{2}\right)^{\alpha} (N-1)^{1-\alpha} \quad (2.7)$$

The changes in Renyi's entropy, Tsalli's entropy, Normalized Tsalli's entropy and Kapur's entropy under deformed specification (2.2) and (2.5) can be evaluated as:

III.

Theorem 3.1: Kapur's Entropy (1.1) is stable under the above deformed specification (2.2) and (2.5).

Proof. $$H^{(K)}_{\alpha,\beta}(p) = \frac{1}{\beta-\alpha}\left[\ln\left(\sum_{i=1}^{N}(p_i)^{\alpha}\right) - \left(\ln \sum_{i=1}^{N} p_i^{\beta}\right)\right] \quad (3.1.1)$$

Using (2.3), (2.4), (2.6) and (2.7) in (6.1), we get

(i) For $0 < \alpha, \beta < 1$: $H_{\alpha}^{(K)}(p) = 0$ (3.1.2)

$$H^{(K)}_{\alpha,\beta}(p') = \ln\left[\left(1-\frac{\delta}{2}\right)^{\alpha} + \left(\frac{\delta}{2}\right)^{\alpha}(N-1)^{1-\alpha}\right] \quad (3.1.3)$$

$$H^{(K)}_{\beta}(p) = 0 \quad (3.1.4)$$

$$H^{(K)}_{\beta}(p') = \ln\left[\left(1-\frac{\delta}{2}\right)^{\beta} + \left(\frac{\delta}{2}\right)^{\beta}(N-1)^{1-\beta}\right] \quad (3.1.5)$$

Thus, $$\left|\frac{H^{(K)}_{\alpha,\beta}(p) - H^{(K)}_{\alpha,\beta}(p')}{H^{(K)}_{\alpha,\beta\,\max}}\right| = \frac{1}{\beta-\alpha}$$

$$\left|\frac{\ln\left[\left(1-\frac{\delta}{2}\right)^{\alpha} + \left(\frac{\delta}{2}\right)^{\alpha}(N-1)^{1-\alpha}\right]}{\ln N} \quad \frac{\ln\left[\left(1-\frac{\delta}{2}\right)^{\beta} + \left(\frac{\delta}{2}\right)^{\beta}(N-1)^{1-\beta}\right]}{\ln N}\right| \quad (3.1.6)$$

$$\text{As } N \to \infty, \left|\frac{H^{(K)}_{\alpha,\beta}(p) - H^{(K)}_{\alpha,\beta}(p')}{H^{(K)}_{\alpha,\beta\,\max}}\right| \to 0 \quad (3.1.7)$$

(ii) For $\alpha, \beta > 1$: $H_{\alpha}^{(K)}(p) = \ln(N-1)$ (3.1.8)

$$H^{(K)}_{\alpha}(p') = \ln\left[\left(\frac{\delta}{2}\right)^{\alpha} + \left(1-\frac{\delta}{2}\right)^{\alpha}(N-1)^{1-\alpha}\right] \quad (3.1.9)$$

$$H_{\beta}^{(K)}(p) = \ln(N-1) \quad (3.1.10)$$

$$H^{(K)}_{\beta}(p') = \ln\left[\left(\frac{\delta}{2}\right)^{\beta} + \left(1-\frac{\delta}{2}\right)^{\beta}(N-1)^{1-\beta}\right] \quad (3.1.11)$$

Thus, $$\left|\frac{H^{(K)}_{\alpha,\beta}(p) - H^{(K)}_{\alpha,\beta}(p')}{H^{(K)}_{\alpha,\beta\,\max}}\right| =$$

$$\ln\left|\frac{(N-1) - \frac{1}{\beta-\alpha}\ln\left[\left(\frac{\delta}{2}\right)^{\alpha} + \left(1-\frac{\delta}{2}\right)^{\alpha}(N-1)^{1-\alpha}\right]}{\ln N} \quad \frac{\ln\ (N-1) - \frac{1}{\beta-\alpha}\ln\left[\left(\frac{\delta}{2}\right)^{\beta} + \left(1-\frac{\delta}{2}\right)^{\beta}(N-1)^{1-\beta}\right]}{\ln N}\right| \quad (3.1.12)$$

$$\text{As } N \to \infty, \left|\frac{H^{(K)}_{\alpha,\beta}(p) - H^{(K)}_{\alpha,\beta}(p')}{H^{(K)}_{\alpha,\beta\,\max}}\right| \to 0 \quad (3.1.13)$$

From (3.1.7) and (3.1.13), we conclude that Kapur's entropy (1.1) is stable.

IV. CONCLUSION

In this paper, we have shown that Kapur's entropy is stable and hence gives more convergent experimental observations. Also, it is used for further statistical generalizations.

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